Algebras for Regular Relations

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Structure meets Power
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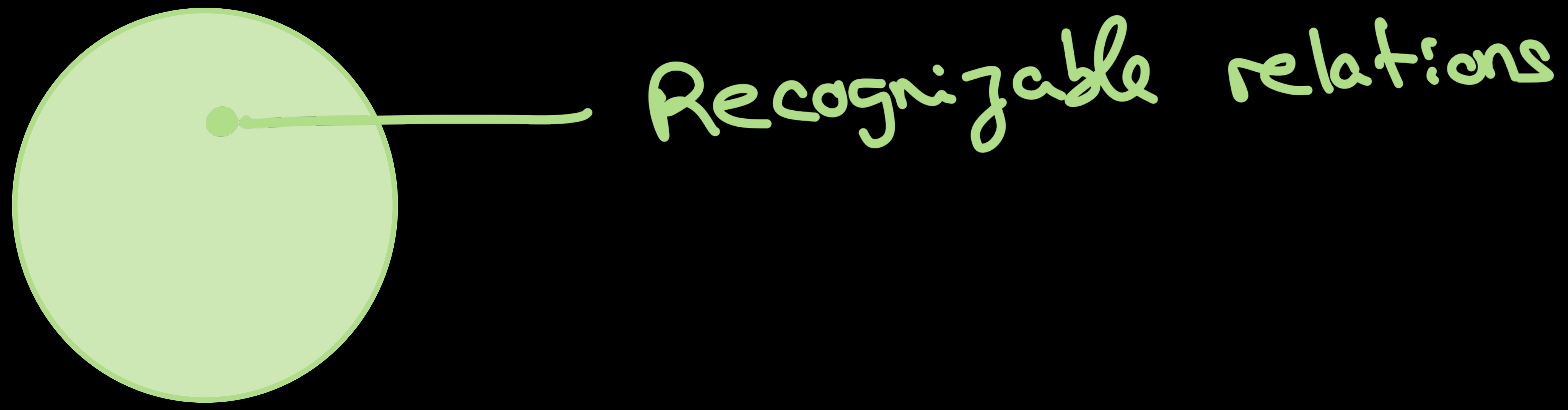
Work in progress !

Relations over Words

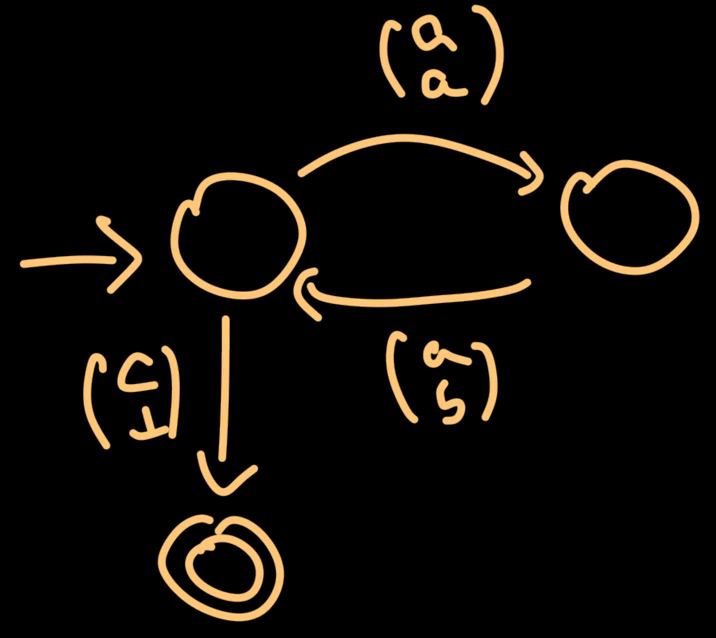


Relations over Words

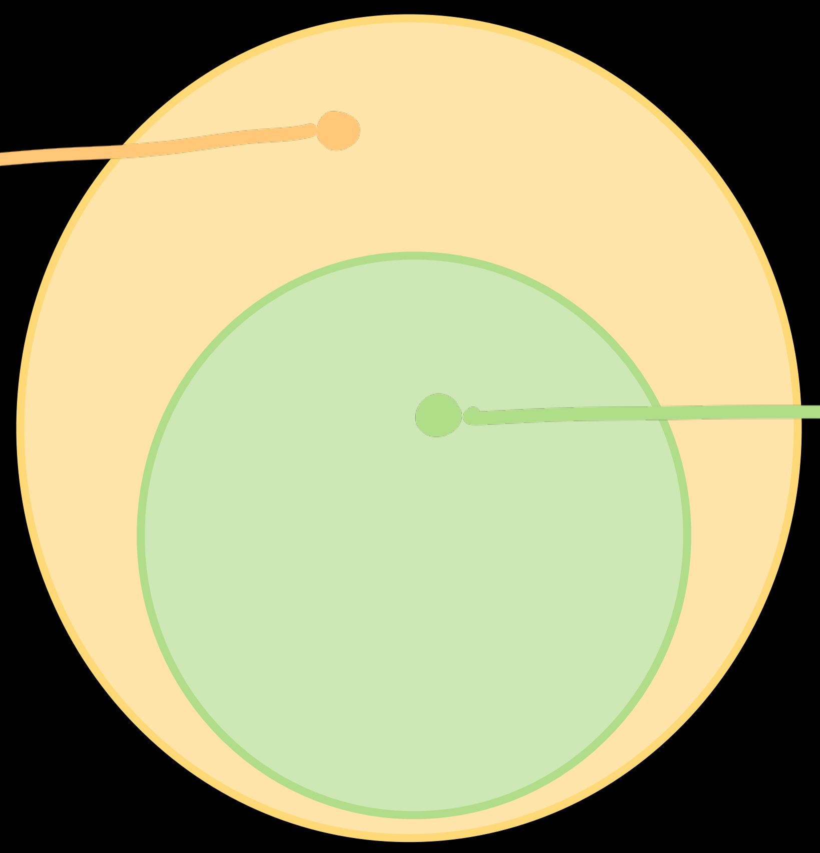
Regular relations

[Eilenberg, Elgot, Shepherdson '69]

u :	u_1	...	u_n	u_{n+1}	...
v :	v_1	...	v_n	$\perp$	$\perp$

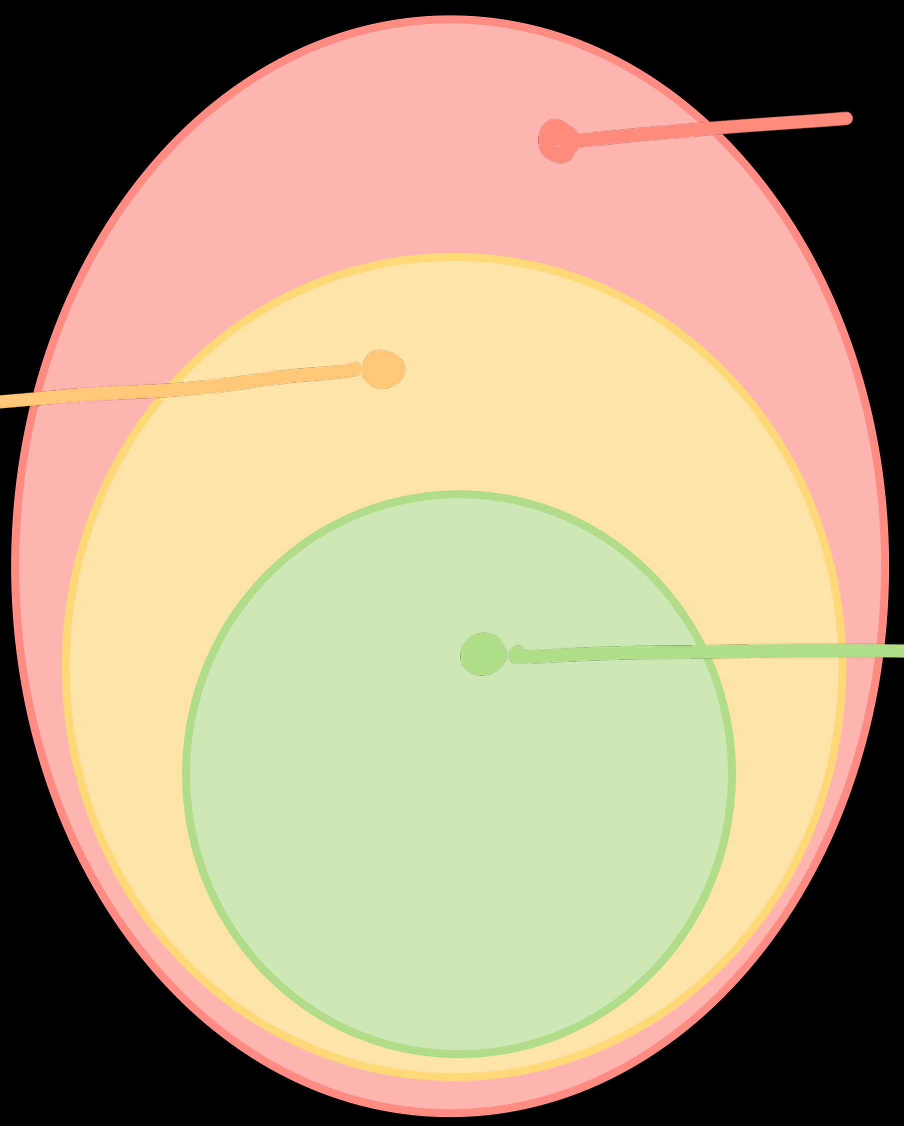


accepts $\{((aa)^n c, (ab)^n) \mid n \in \mathbb{N}\}$



Recognizable relations

Relations over Words

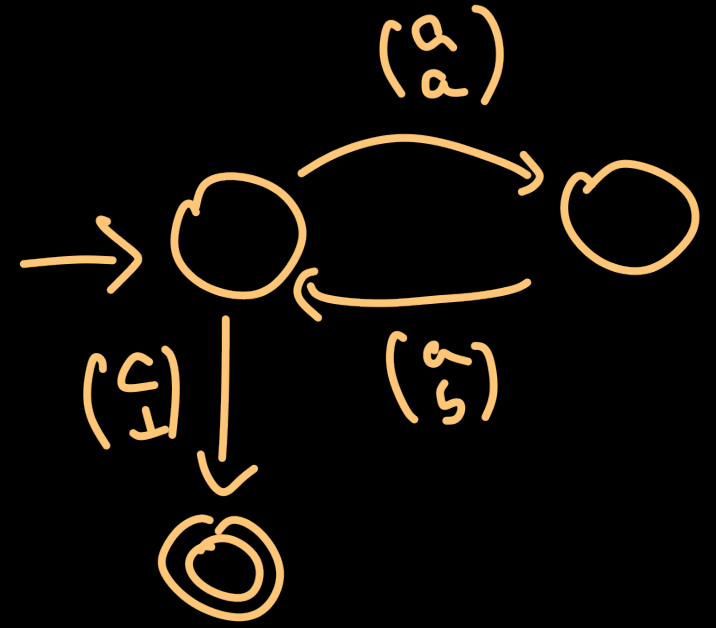


Rational relations
[Rabin & Scott, '59]

Regular relations
[Eilenberg, Elgot, Shepherdson '69]

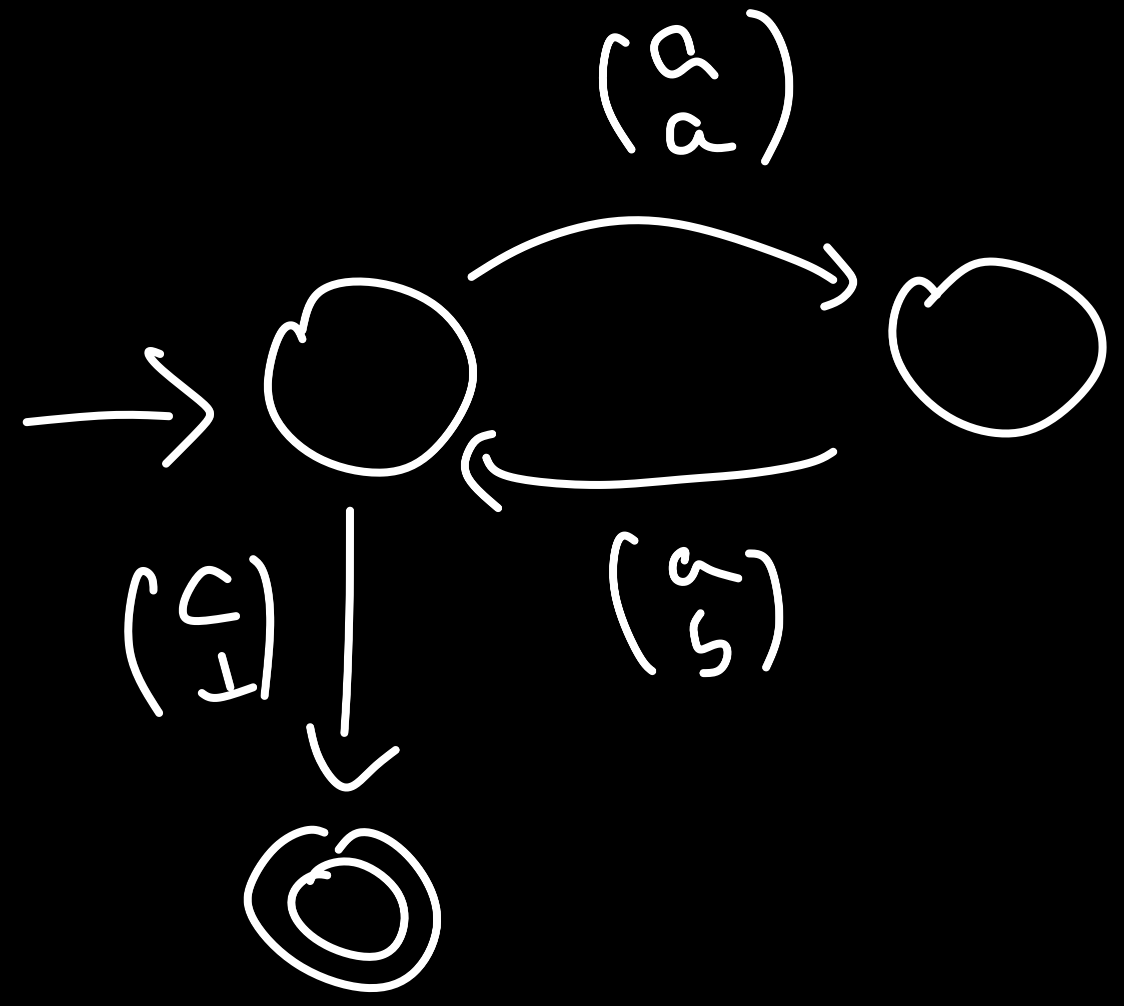
u :	u_1	...	u_n	u_{n+1}	...
v :	v_1	...	v_n	$\perp$	$\perp$

Recognizable relations



accepts $\{((aa)^n c, (ab)^n) \mid n \in \mathbb{N}\}$

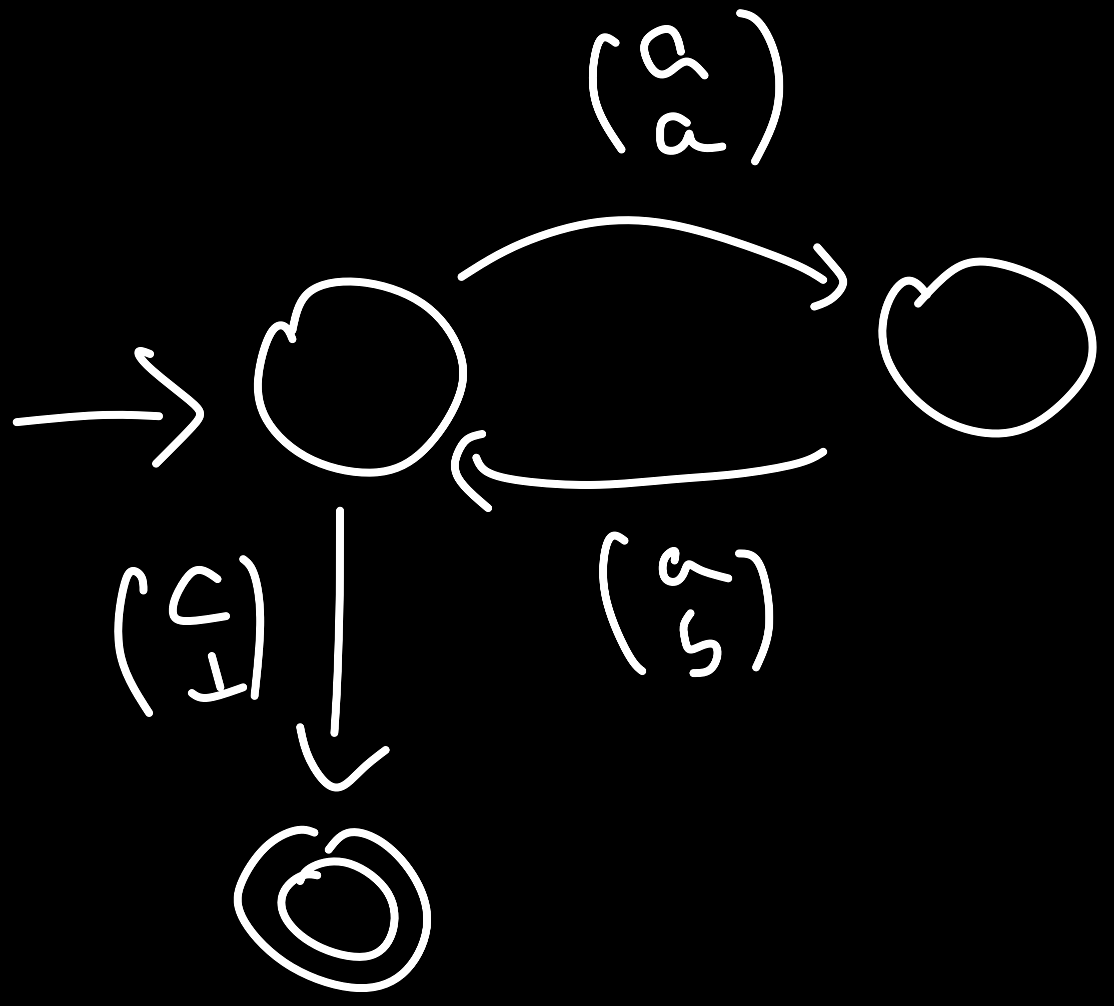
The synchronous model



$$R = \{ ((aa)^n c,\ (ab)^n) \mid n \in \mathbb{N} \} \subseteq \Sigma^+ \times \Sigma^+$$

$$\hat{R} = \left\{ \left[\begin{pmatrix} a \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right]^n \begin{pmatrix} c \\ \perp \end{pmatrix} \;\middle|\; n \in \mathbb{N} \right\} \subseteq (\Sigma^2_\perp)^+$$

The synchronous model



Models :

$\begin{pmatrix} a & b & a & b & a & a & b \\ a & a & a & a & \perp & \perp & \perp \end{pmatrix}$ ✓ « well-formed »

$\begin{pmatrix} a & a & a \\ a & \perp & b \end{pmatrix}$ ✗

$$R = \{ ((aa)^n c,\ (ab)^n) \mid n \in \mathbb{N} \} \subseteq \Sigma^+ \times \Sigma^+$$

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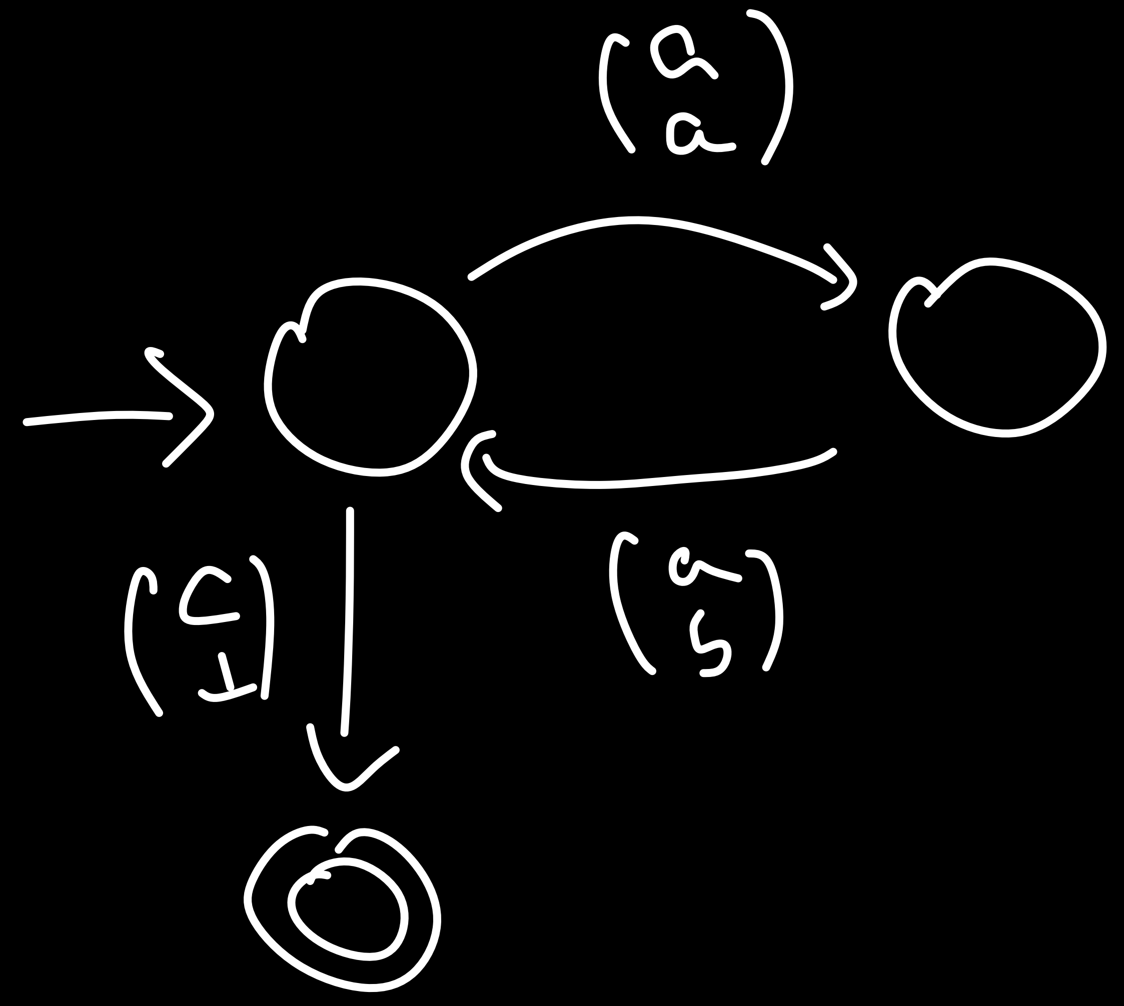

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$\mathcal{F}$ fragment of $\mathrm{MSO}[<, \underbrace{\binom{a}{b}, \binom{a}{\perp}, \binom{\perp}{a}}_{\text{unary}}]$

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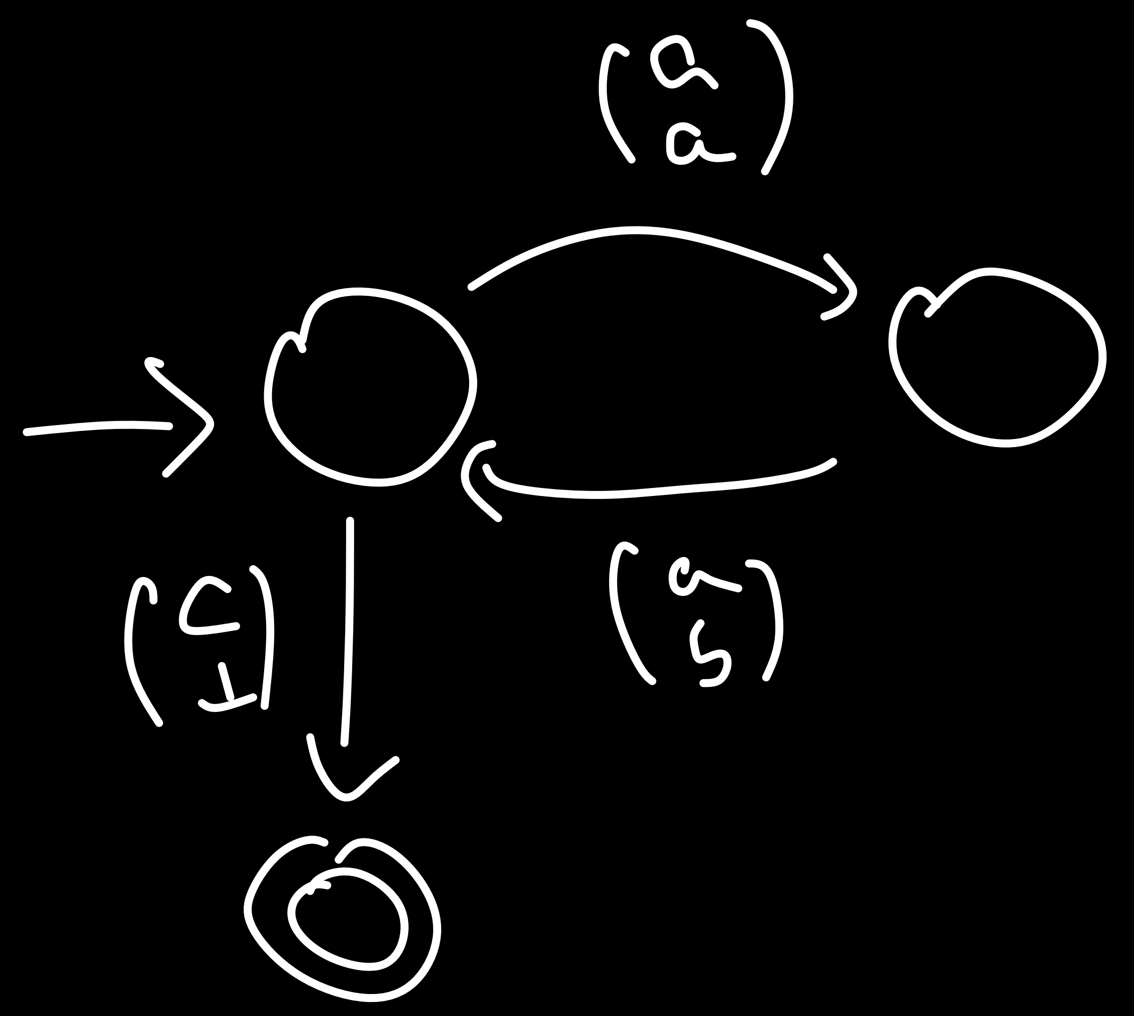

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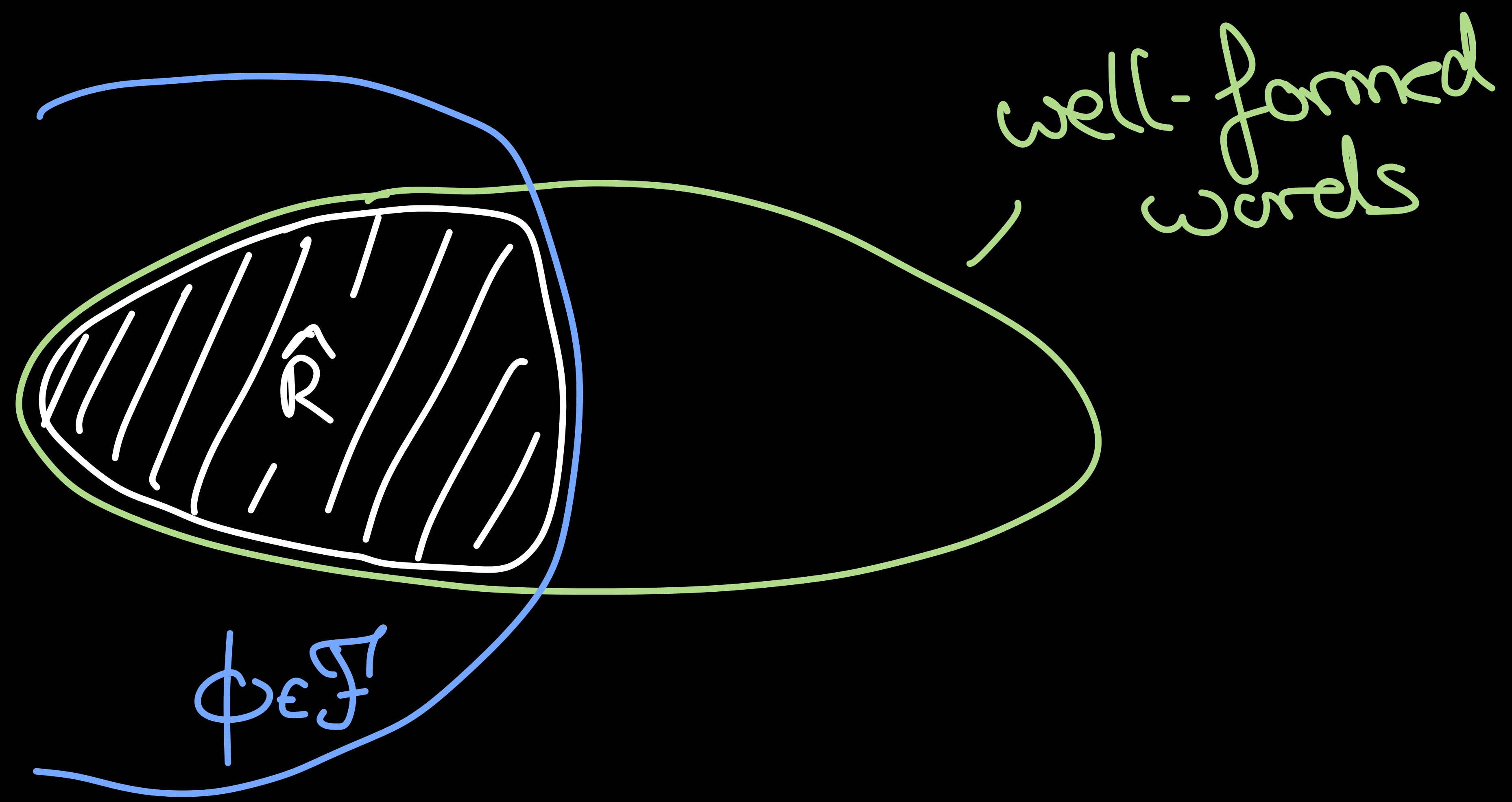
Q° Is $\hat{R}$ expressible in $\mathcal{F}$?

$\forall u \in (\Sigma_\perp^2)^+$, u well-formed

$\Rightarrow u \models \underbrace{\phi}_{\in \mathcal{F}}$ iff $u \in \hat{R}$

The synchronous model



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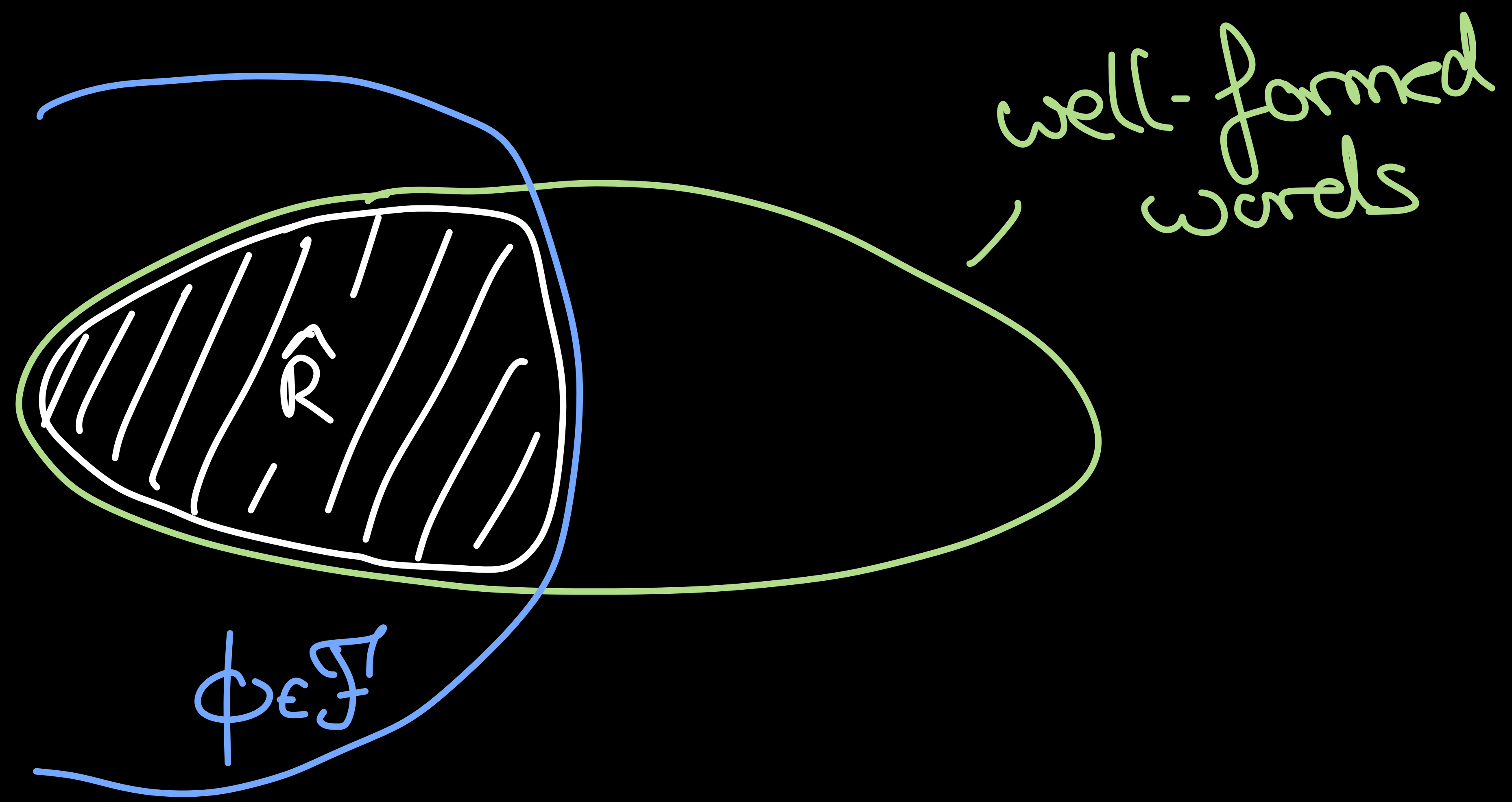

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$\begin{pmatrix} a & a & a \\ a & \perp & b \end{pmatrix}$ ✗

Ex First-order logic

$\hat{R}$ is expressible in FO inside well-formed

$\Longleftrightarrow$

$\hat{R}$ is expressible in FO inside $(\Sigma_\perp^2)^+$

Q° Is $\hat{R}$ expressible in $\mathcal{F}$?

$\forall u \in (\Sigma_\perp^2)^+$, u well-formed

$\Rightarrow$ $u \models \underset{\in \mathcal{F}}{\phi}$ iff $u \in \hat{R}$

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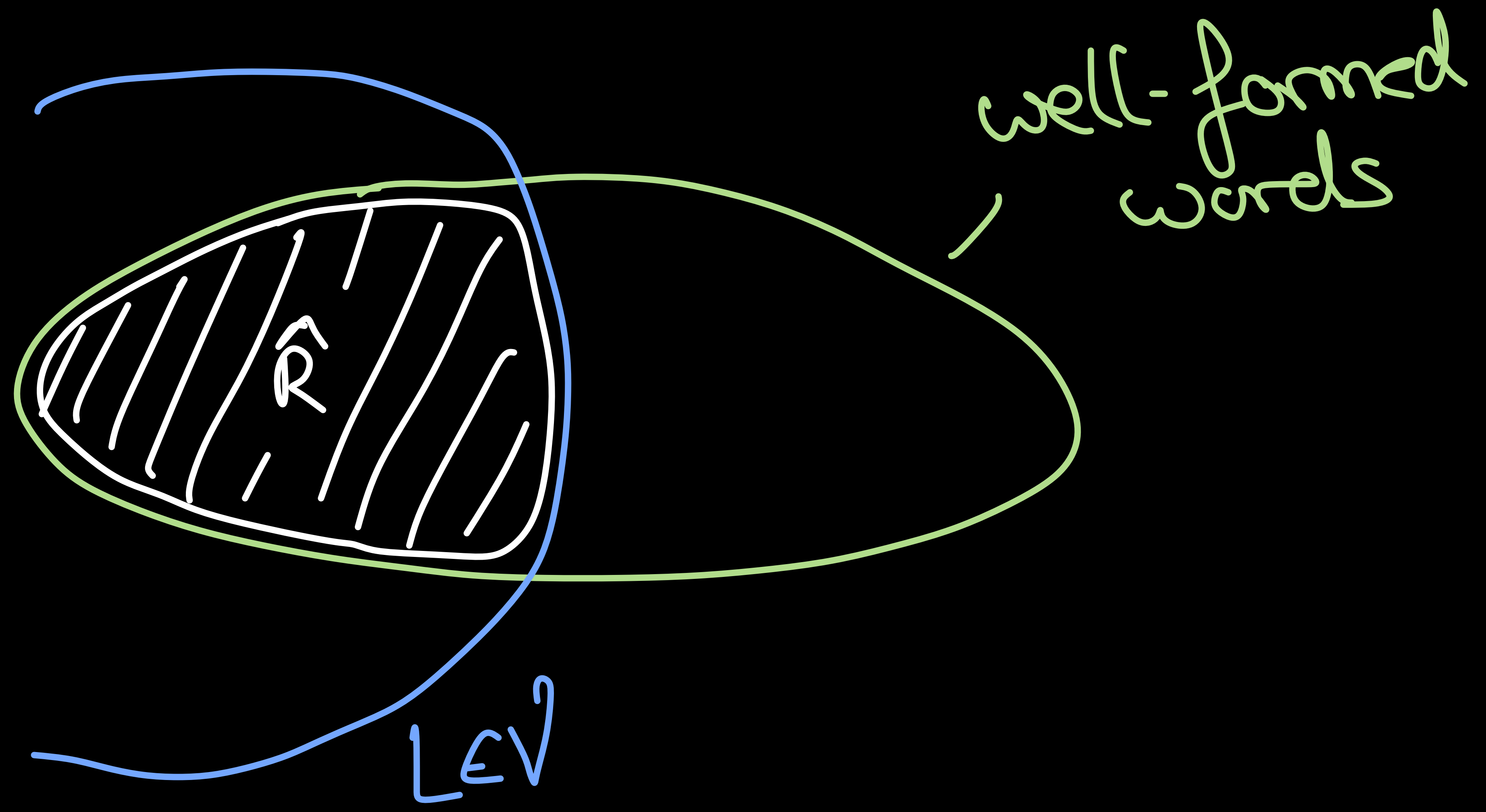

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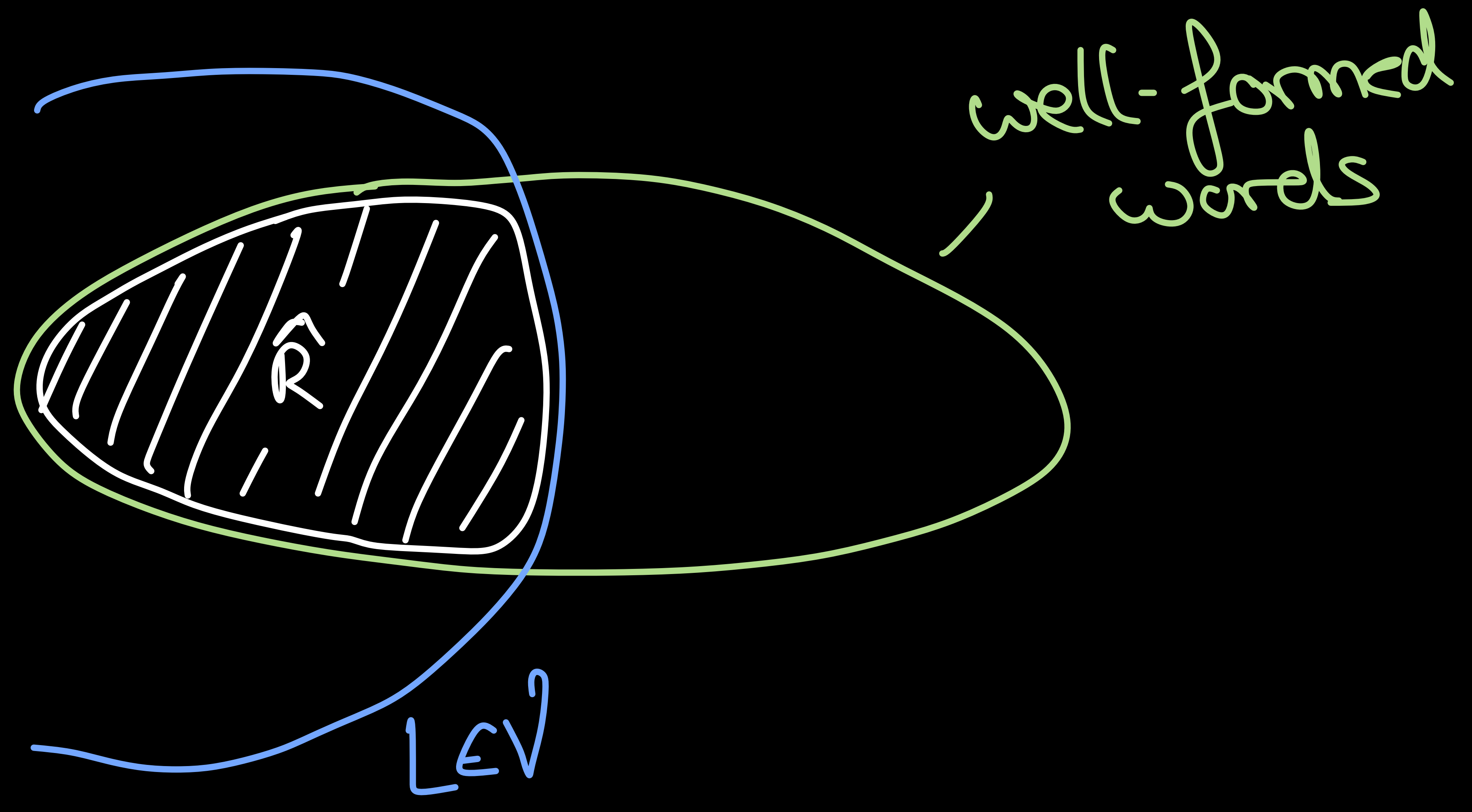
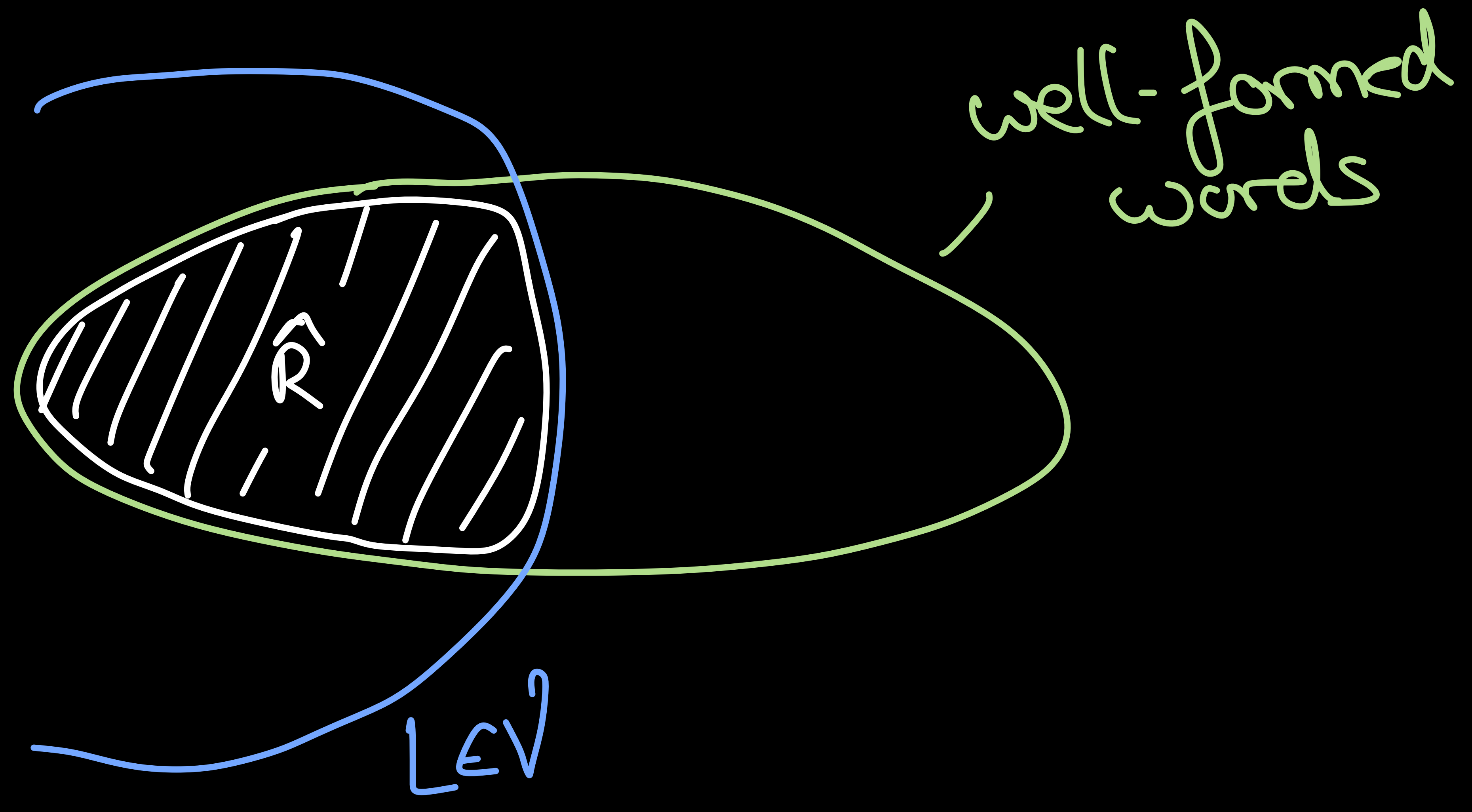


$$\mathcal{V} \text{ pseudovariety of regular languages} \underset{\text{recognizability}}{\overset{\text{synt. semigroup}}{\rightleftarrows}} \mathbf{V} \text{ pseudovariety of finite semigroups}$$

Nota°: $\mathcal{V}^{\ell}_{sync} \mathrel{\hat{=}} \{ \hat{R} \mid \exists L \in \mathcal{V},\ \hat{R} = L \cap \text{well-formed} \}$

The synchronous model



$\mathcal{V}$ pseudovariety of regular languages $\underset{\text{recognizability}}{\overset{\text{synt. semigroup}}{\rightleftarrows}}$ **V** pseudovariety of finite semigroups

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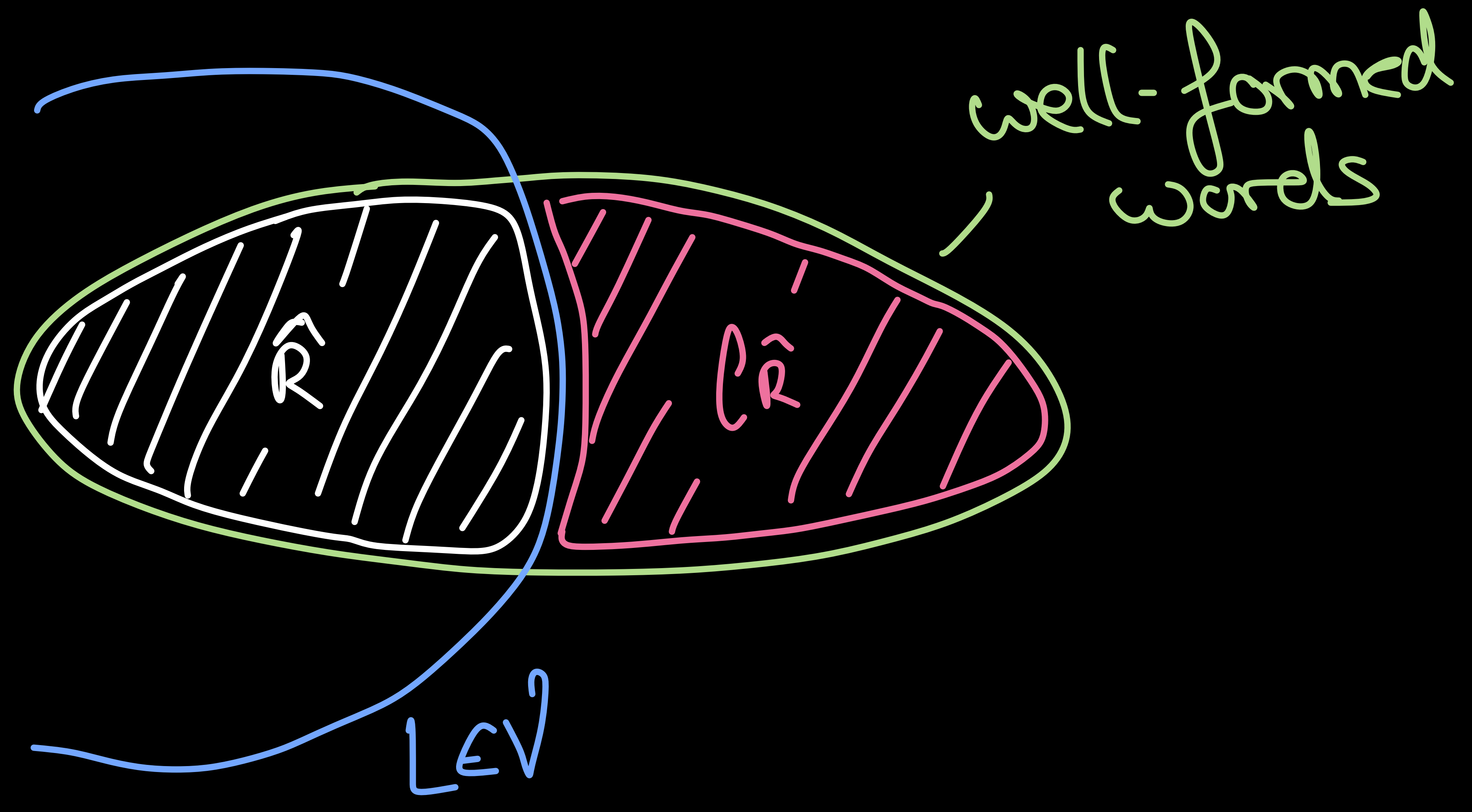
Ex $\mathcal{V}$ = commutative languages

$R = \{ (u,v) \mid |v| - |u| \text{ is even} \}$

$\hat{R} = \{ u \in (\Sigma^{?}_{\perp})^{+} \mid \text{even nb. of } \binom{a}{\perp} \text{ or } \binom{\perp}{a} \}$
$\cap$ well-formed
$\in \mathcal{V}^{\ell}_{sync}$

The synchronous model



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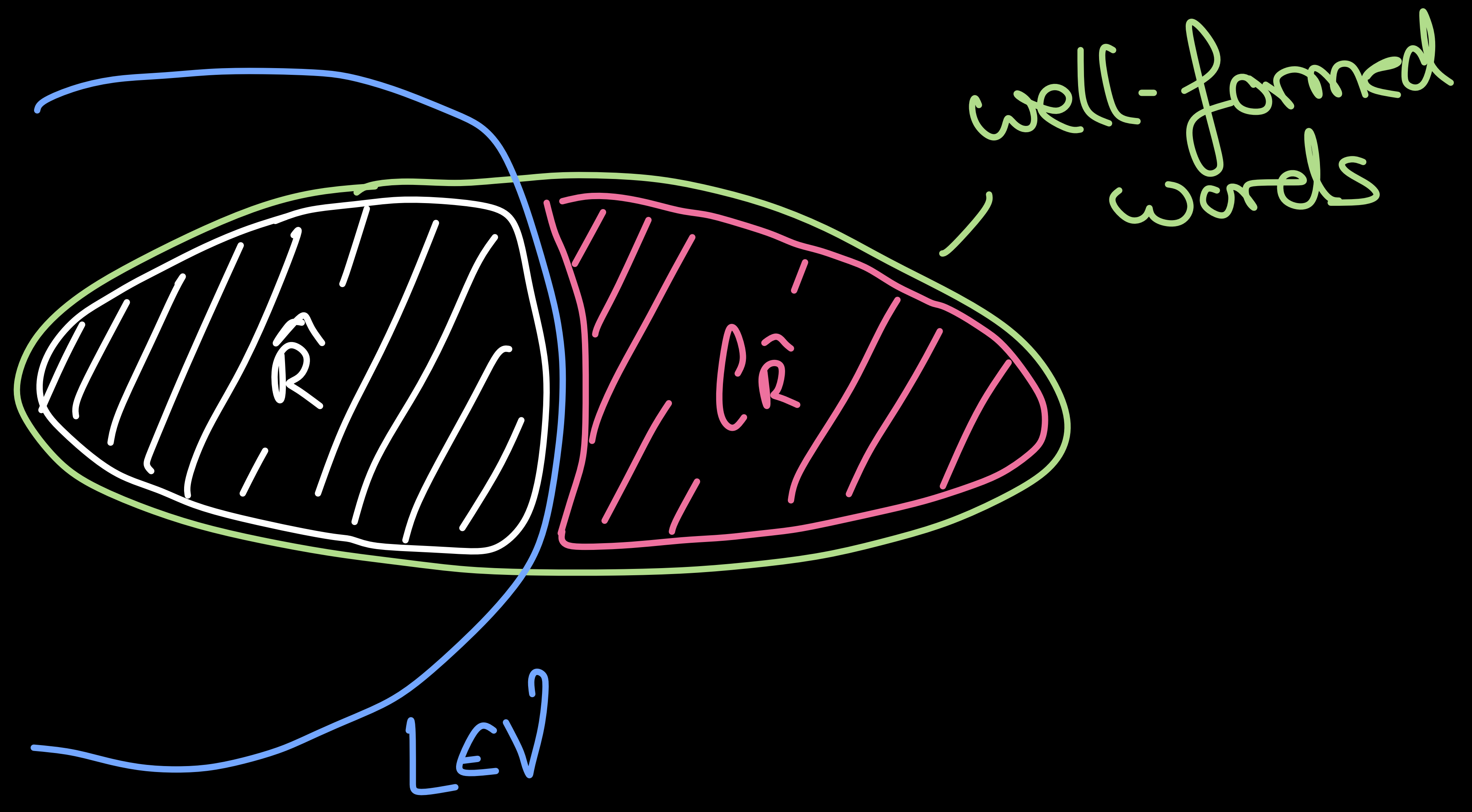

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$\cap$ well-formed
$\in \mathcal{V}^{\ell}_{sync}$

Prop $\mathcal{V}$-separation is decidable
$\Rightarrow$ $\mathcal{V}^{\ell}_{sync}$-membership is decidable

Synchronous Algebras

Well-formed words: $\begin{pmatrix} a & a & c \\ b & a & b \end{pmatrix}$, $\begin{pmatrix} a & \perp \\ c & c \end{pmatrix}$, $\begin{pmatrix} \perp & \perp & \perp \\ a & b & a \end{pmatrix}$ + symmetric

$l/l \to l/l$ $\quad$ $l/l \to b/l$ $\quad$ $b/l \to b/l$ $\quad$ $l/l \to l/b$

$l/b \to l/b$

Synchronous Algebras

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$\ell/\ell \to \ell/\ell$, $\ell/\ell \to b/\ell$, $b/\ell \to b/\ell$, $\ell/\ell \to \ell/b$, $\ell/b \to \ell/b$

Concatenation: $\underbrace{\begin{pmatrix} a & \cdots & c \\ b & \cdots & d \end{pmatrix}}_{\ell/\ell \to \ell/\ell} \cdot \underbrace{\begin{pmatrix} e & \perp & \cdots & \perp \\ f & g & \cdots & h \end{pmatrix}}_{\ell/\ell \to b/\ell}$: $\ell/\ell \longrightarrow b/\ell$

$\underbrace{\begin{pmatrix} \perp & \perp \\ a & b \end{pmatrix}}_{b/\ell \to b/\ell} \cdot \underbrace{\begin{pmatrix} c \\ e \end{pmatrix}}_{\ell/\ell \to \ell/\ell}$ ✗

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$b/\ell \qquad b/\ell$

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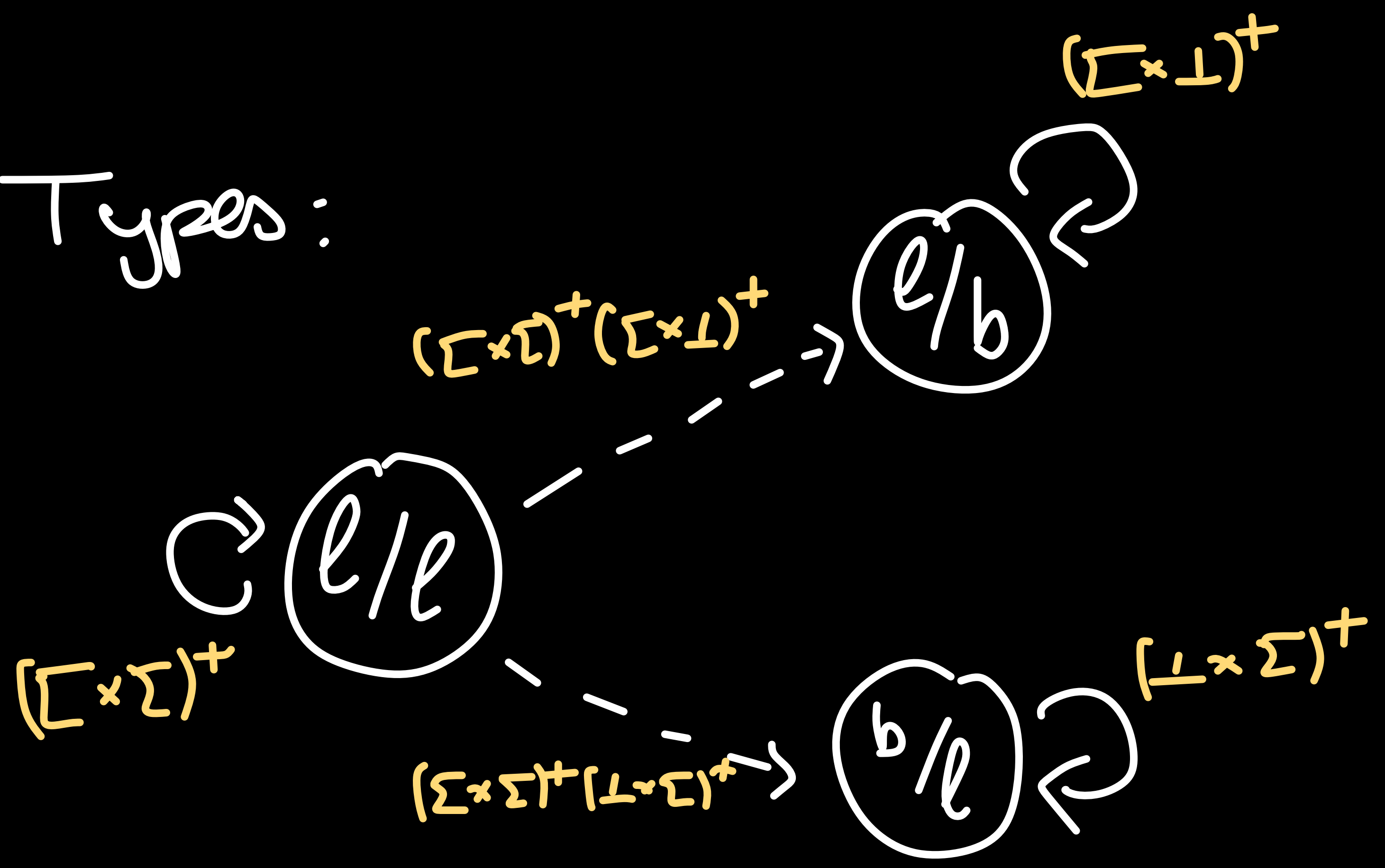
Types:



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$\ell/\ell \to \ell/\ell$ $\ell/\ell \to b/\ell$ $b/\ell \to b/\ell$ $\ell/\ell \to \ell/b$ $\ell/b \to \ell/b$

Concatenation: $\begin{pmatrix} a & \cdots & c \\ b & \cdots & d \end{pmatrix} \cdot \begin{pmatrix} e & \perp & \cdots & \perp \\ f & g & \cdots & h \end{pmatrix} : \ell/\ell \longrightarrow b/\ell$

$\ell/\ell \to \ell/\ell$ $\ell/\ell \longrightarrow b/\ell$

$\begin{pmatrix} \perp & \perp \\ a & b \end{pmatrix} \cdot \begin{pmatrix} c \\ e \end{pmatrix}$ ✗

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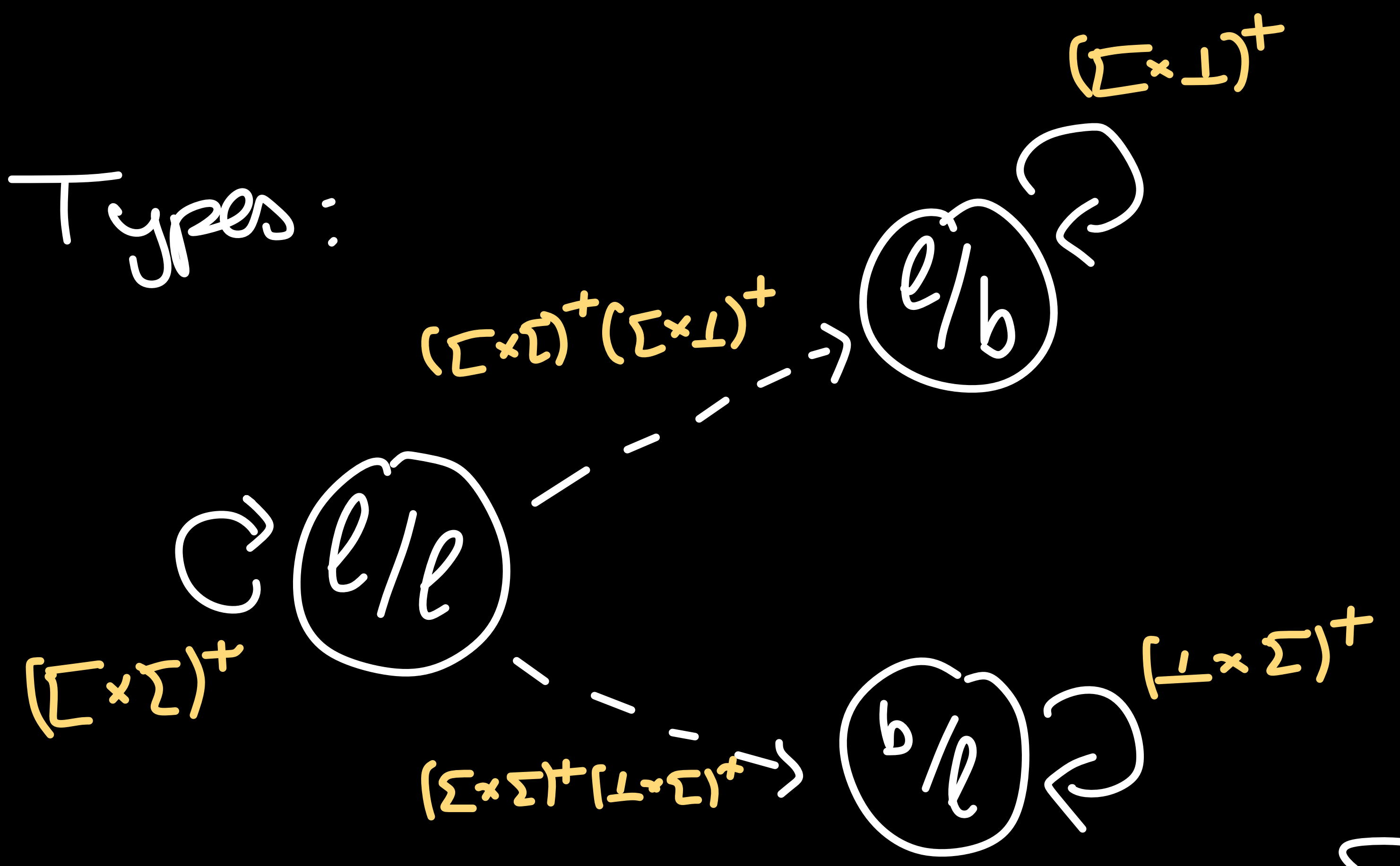
$\ell/\ell \to \ell/\ell$ b/ℓ b/ℓ

Types:



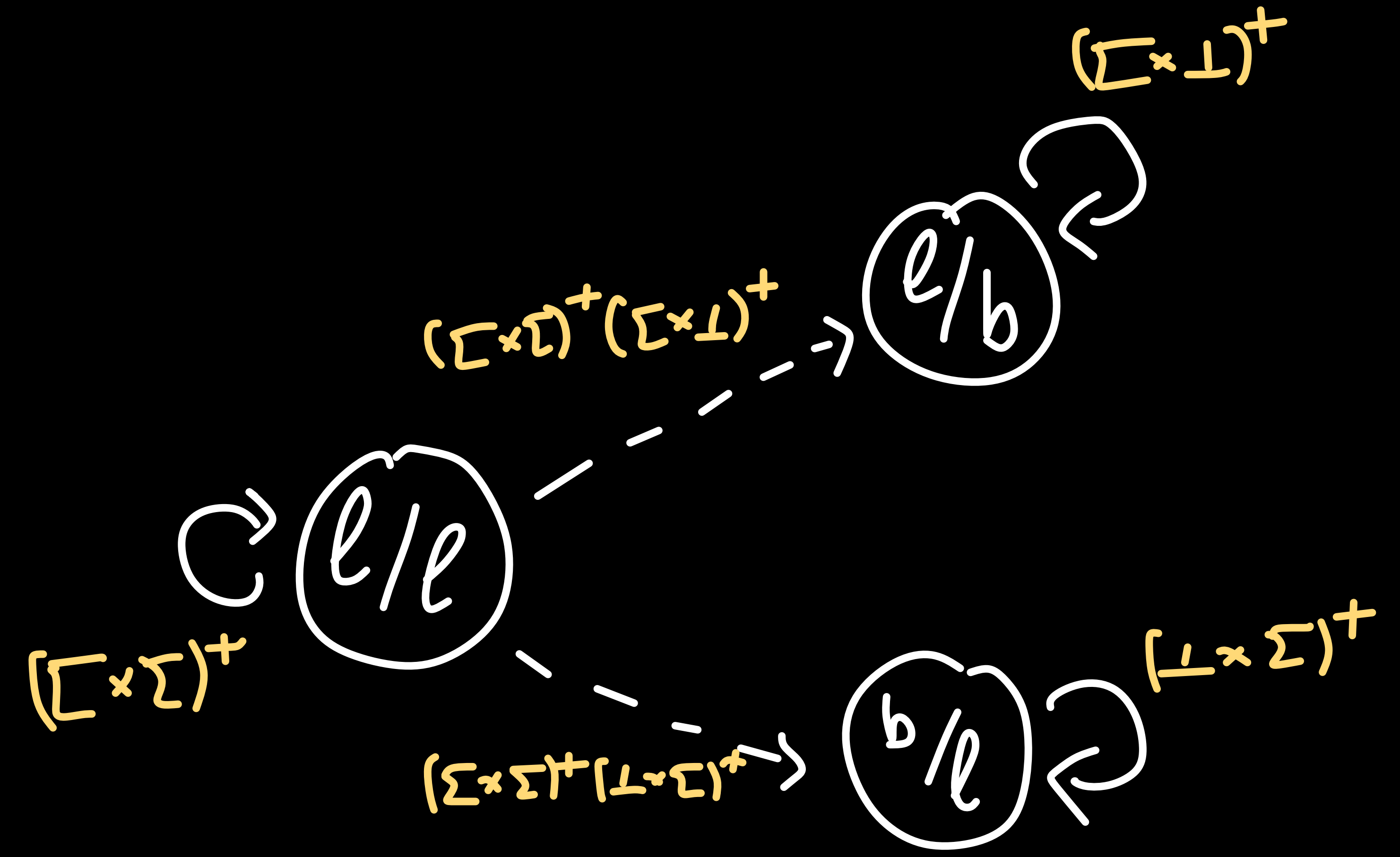
Synchronous algebra:

- sets $A_{\ell/\ell} \to A_{\ell/\ell}$, $A_{\ell/\ell} \to A_{\ell/b}$, $A_{\ell/b} \to A_{\ell/b}$
  $A_{\ell/\ell} \to A_{b/\ell}$, $A_{b/\ell} \to A_{b/\ell}$
- product
- associative

Synchronous Algebras & Monads

Algebra of synchronous words:

$S_2\Sigma$ =



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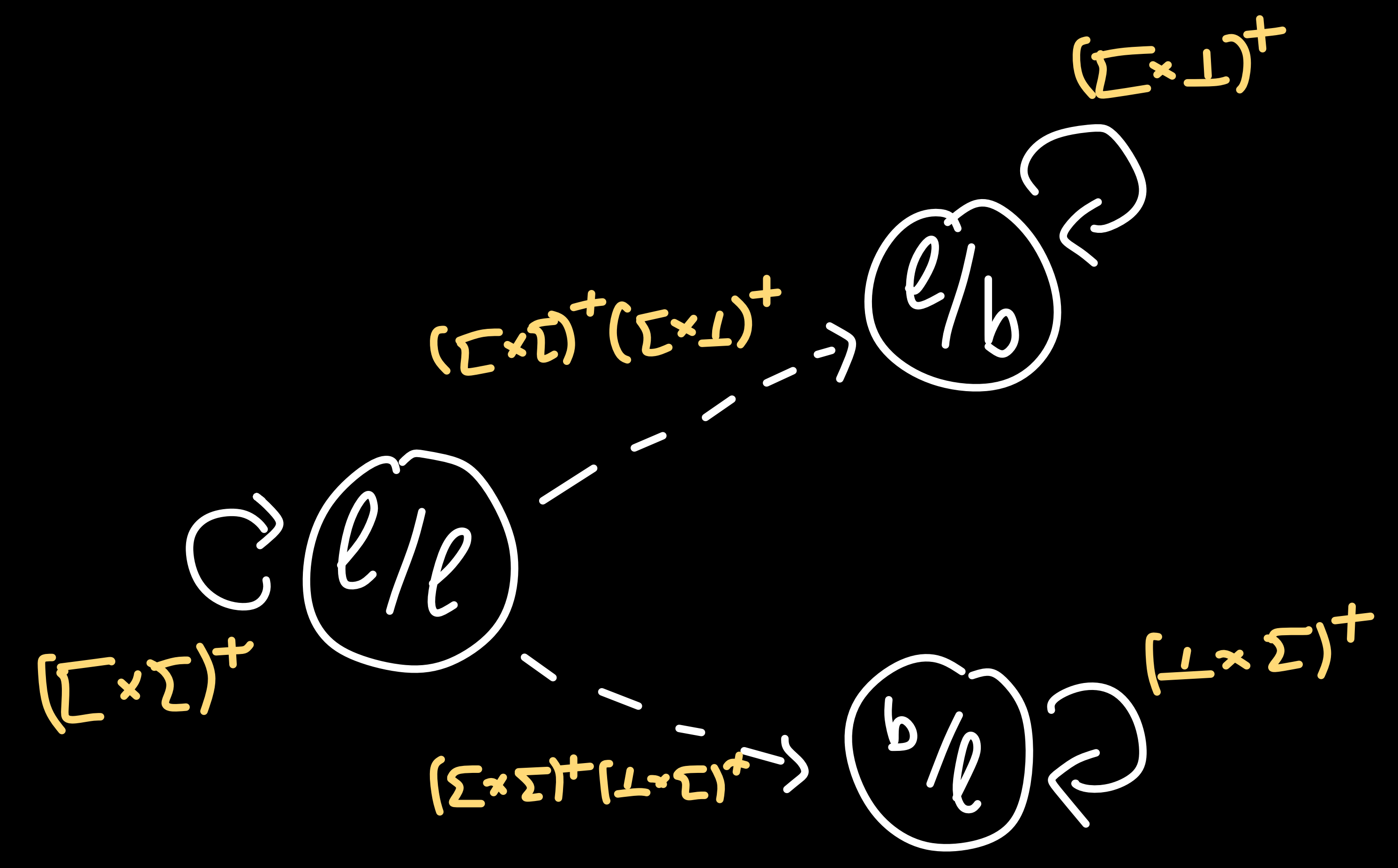
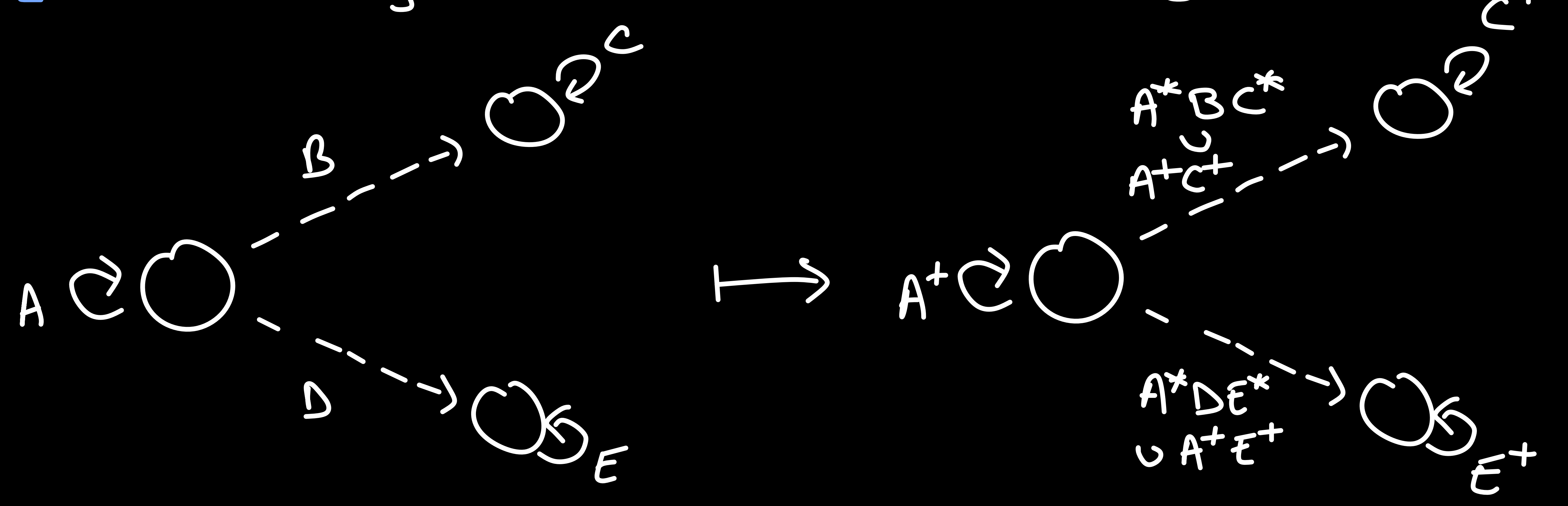


Monad:

S_2 : $\mathrm{Sets}_S \longrightarrow \mathrm{Sets}_S$



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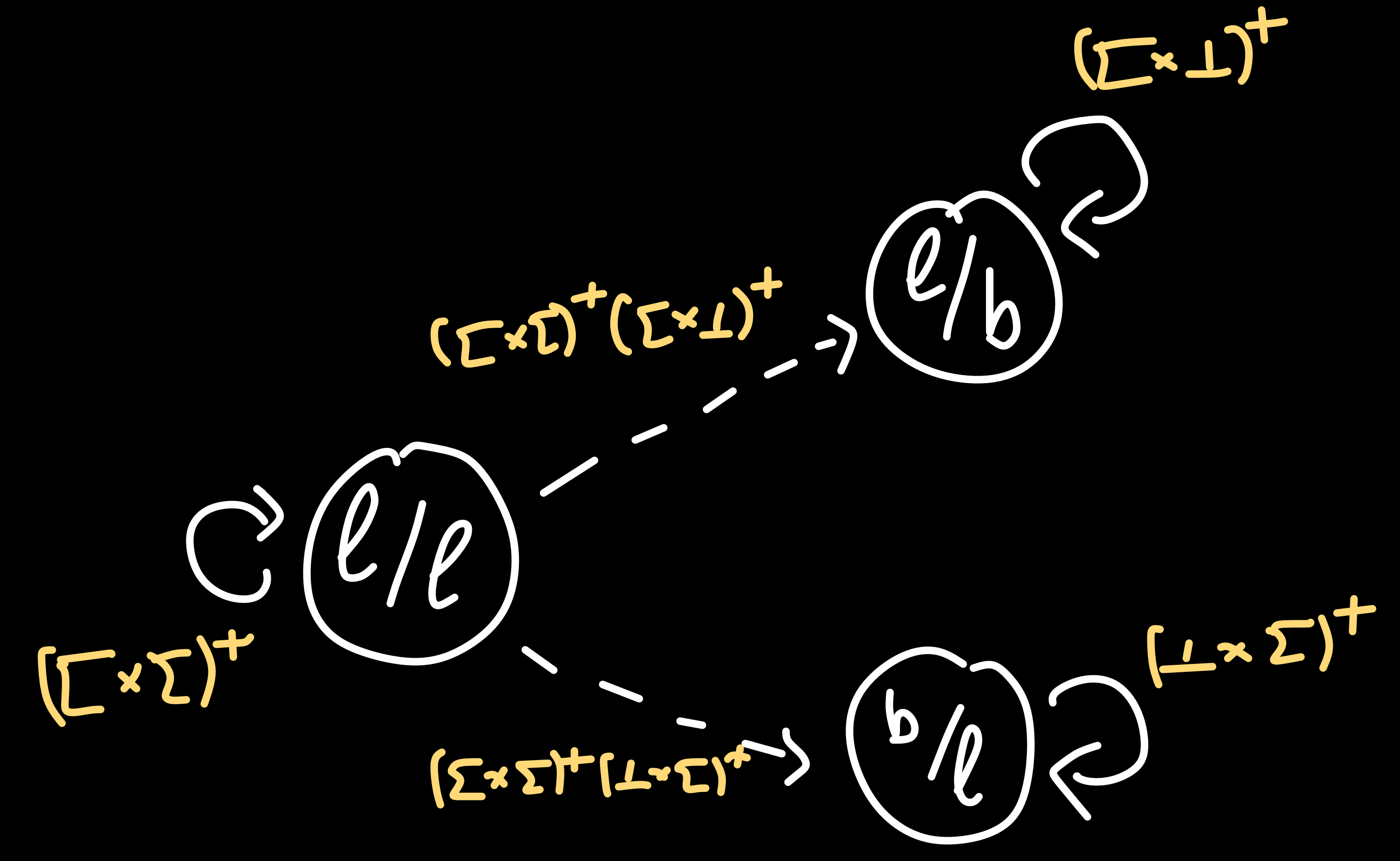
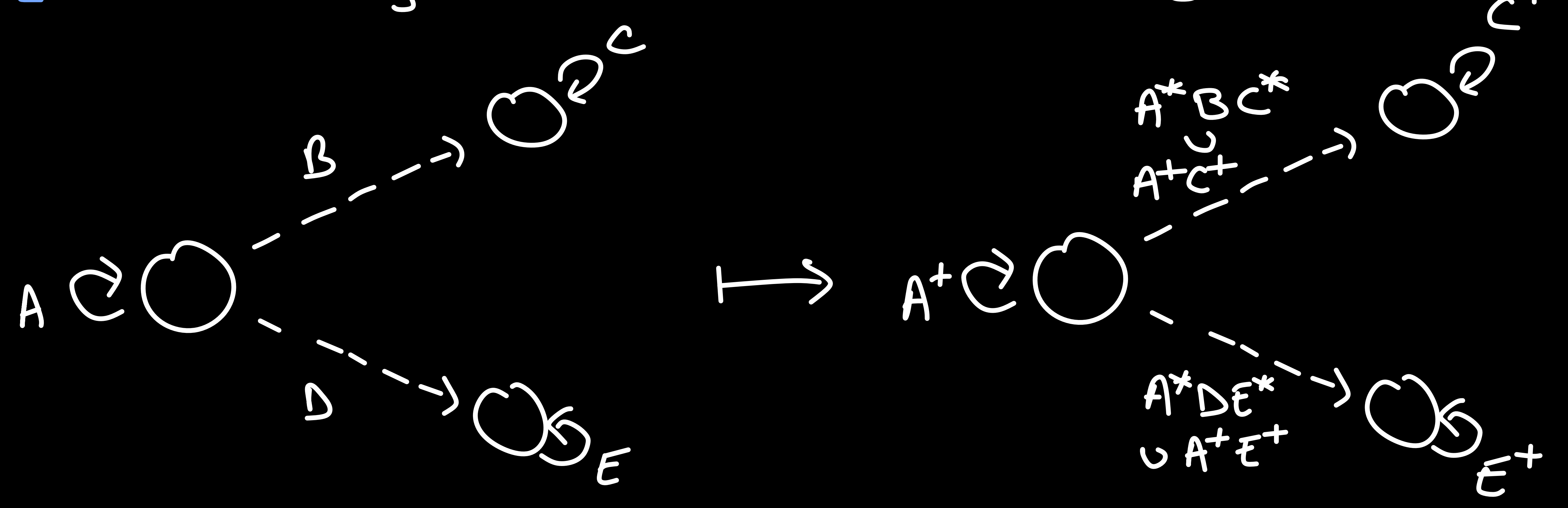


Monad:

$S_2 :$ Sets $\longrightarrow$ Sets



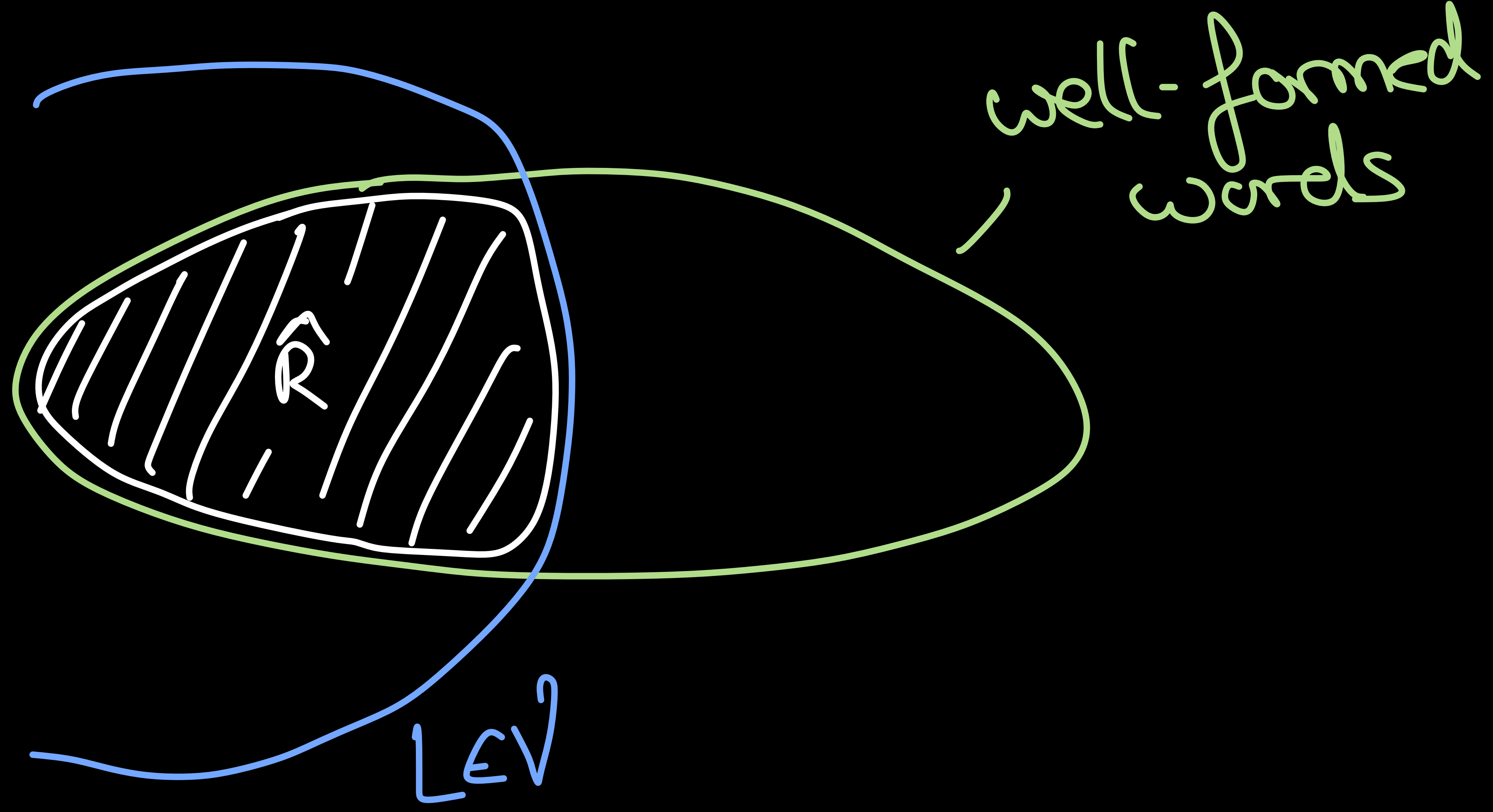
Coro [Bojańczyk, DLT '15]

Existence of syntactic synchronous algebra morphisms.

Thm $R \subseteq S_2\Sigma^*$ is regular IFF it is recognized by a finite synchronous algebra

Algebraic characterizations

Q° $\hat{R} \in \mathcal{V}^{\ell}_{\mathrm{sync}}$?



Algebraic characterizations

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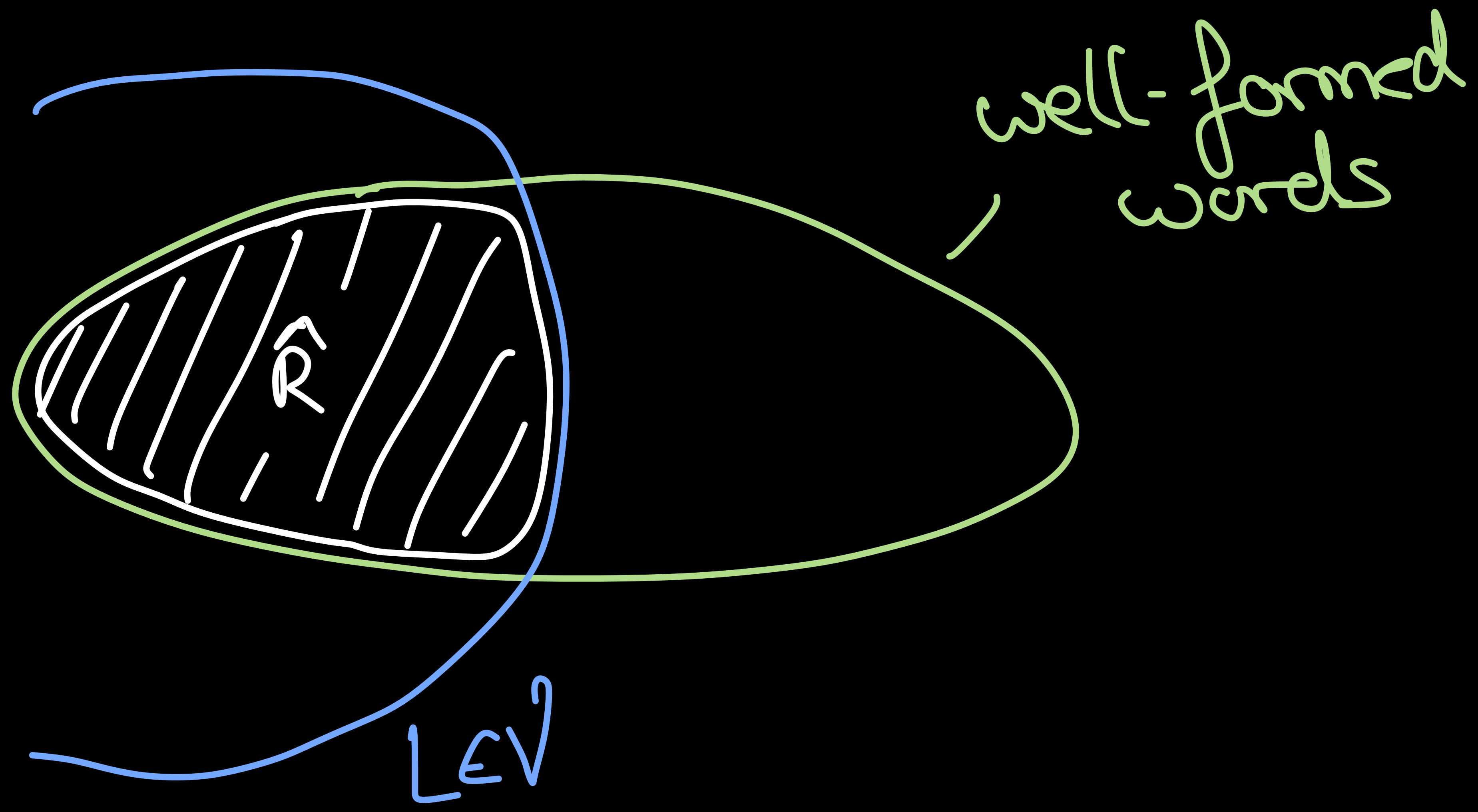
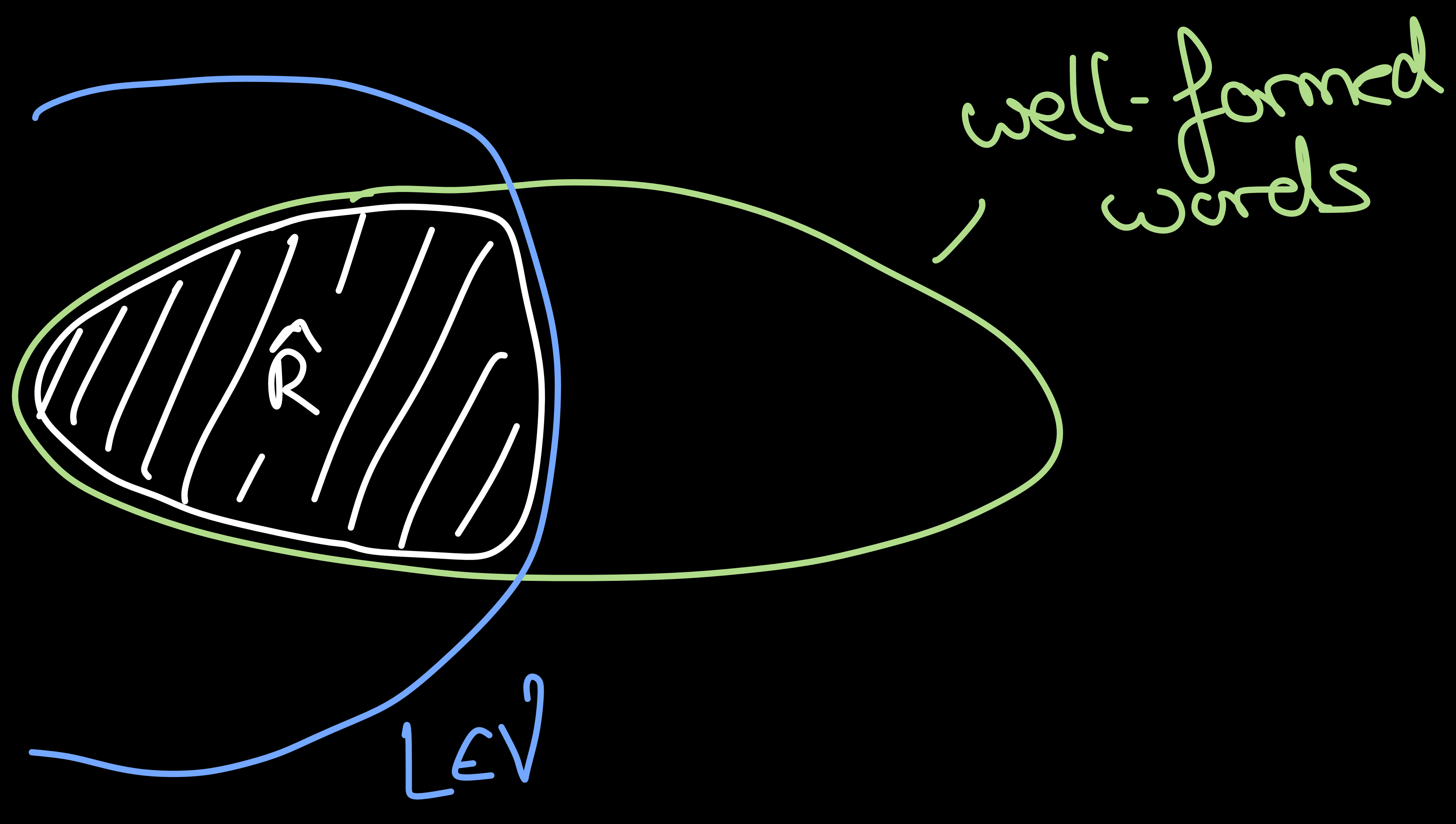


Ex $\mathcal{V}$ = commutative languages

$R = \{ (u,v) \mid |v|-|u| \text{ is even} \} \in \mathcal{V}^{\ell}_{sync}$

Algebraic characterizations



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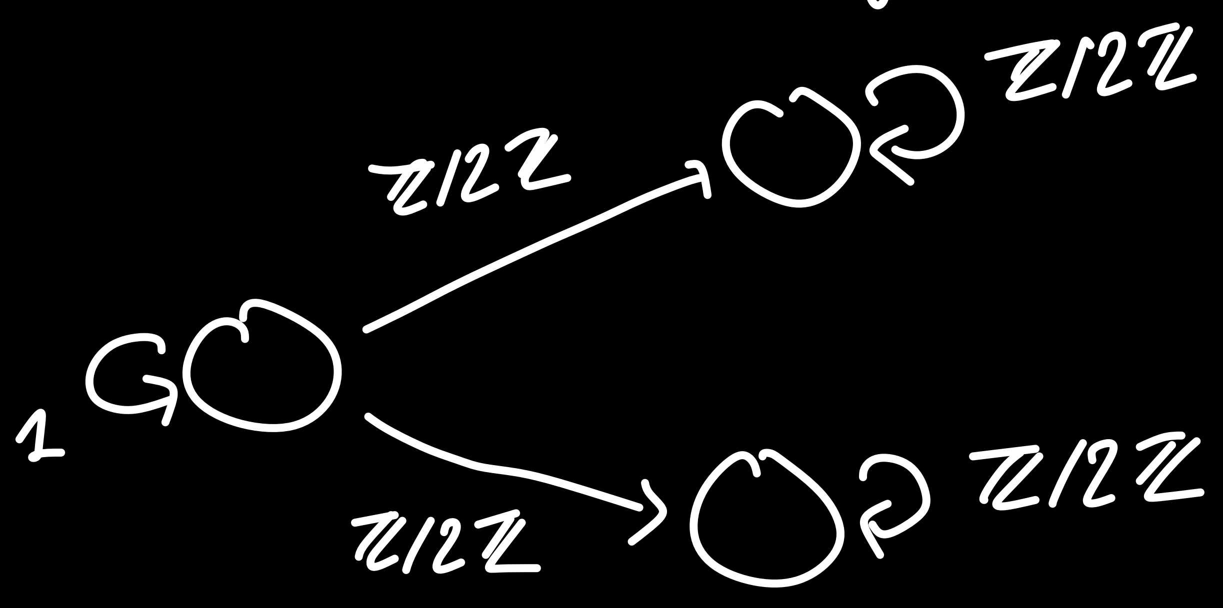
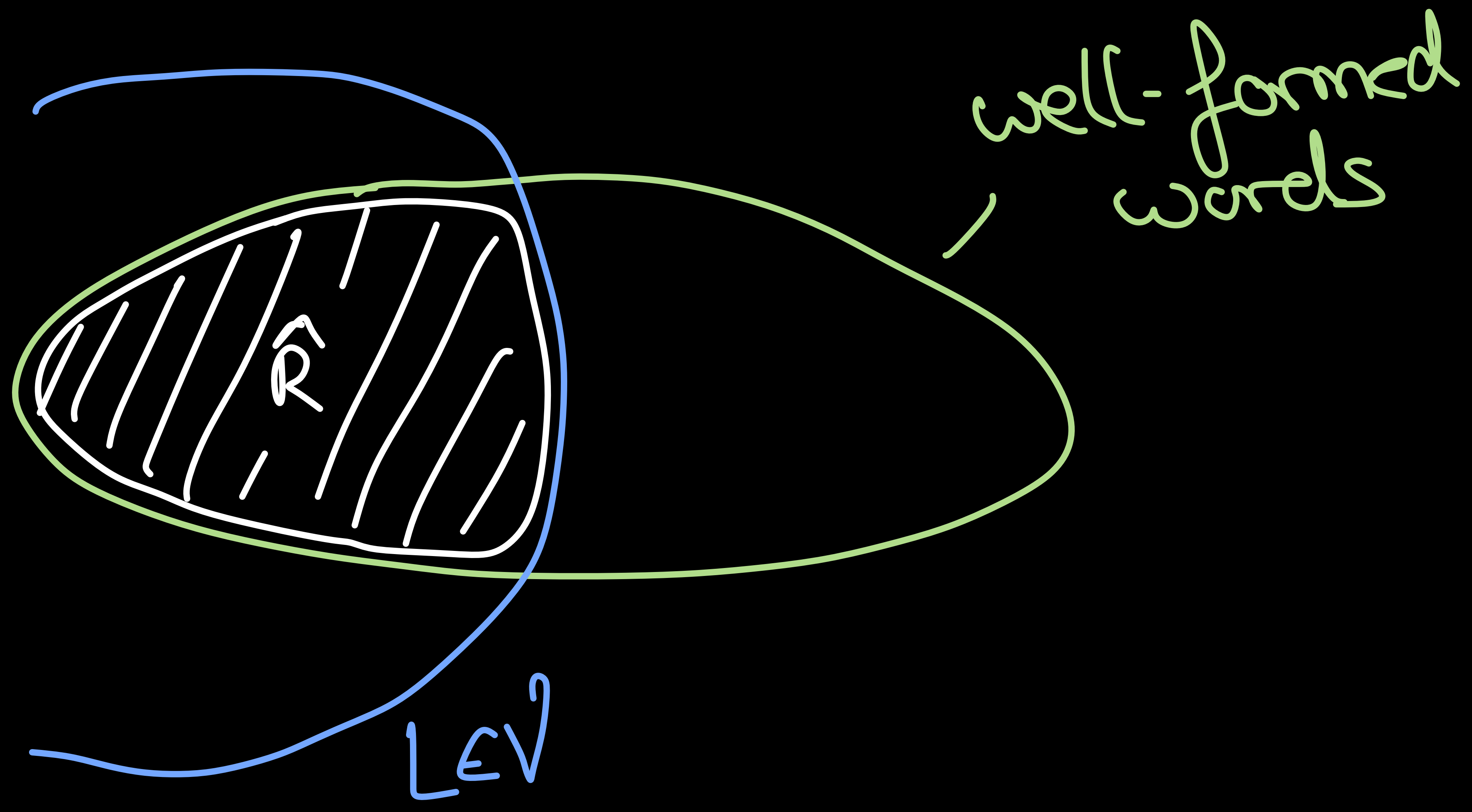
Sync. algebra:



Algebraic characterizations



Q: $\hat{R} \in \mathbb{V}^{\ell}_{sync}$?

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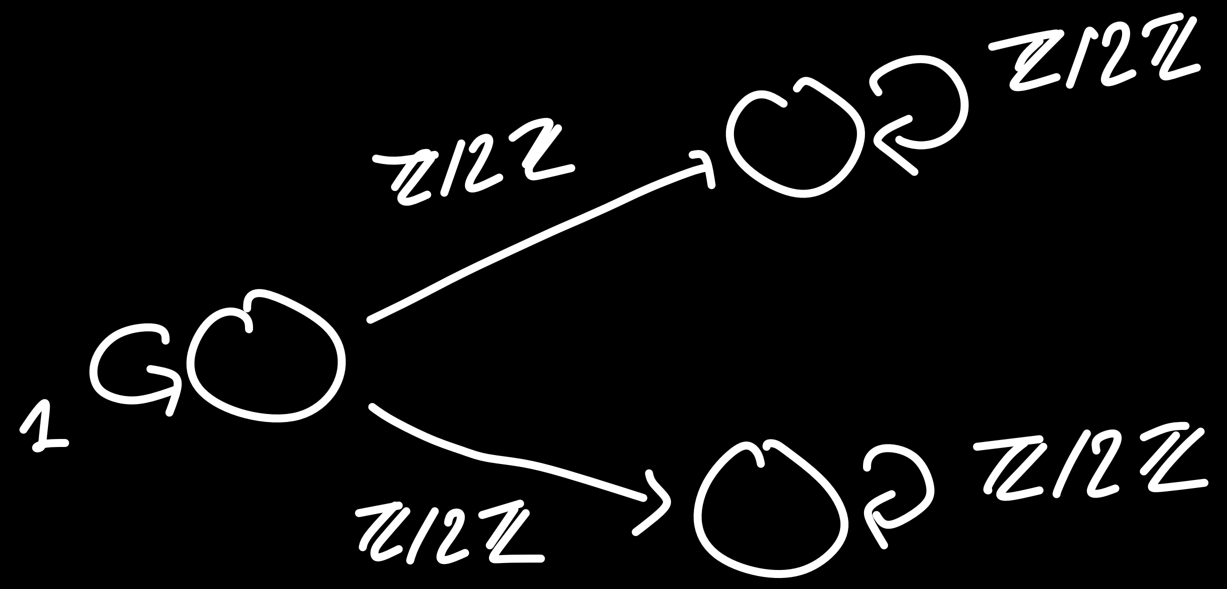
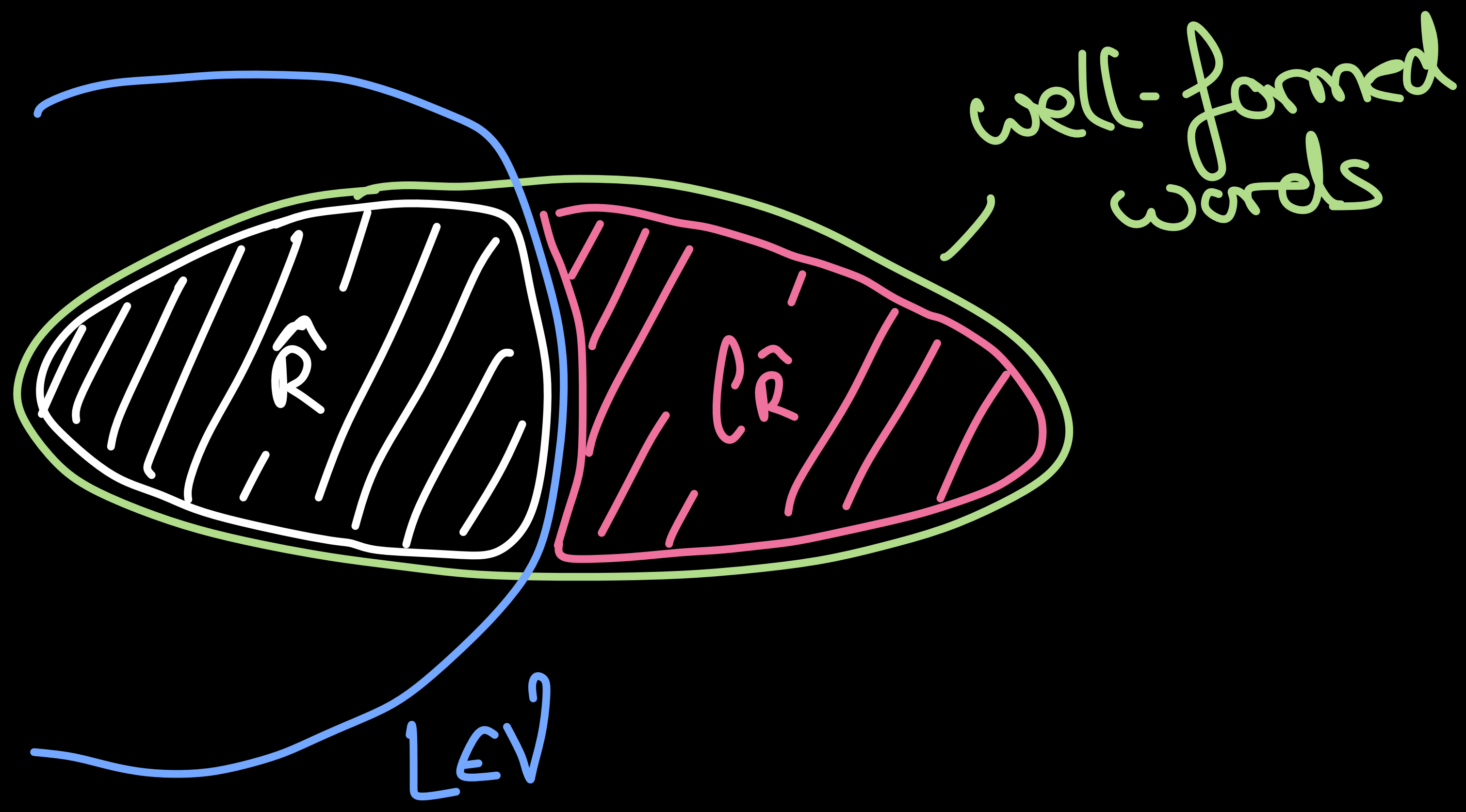
Sync. algebra:



Thm $\hat{R} \in \mathbb{V}^{\ell}_{sync}$ iff the synchronous syntactic algebra of $\hat{R}$ has all underlying semigroups in $\mathbb{V}$.

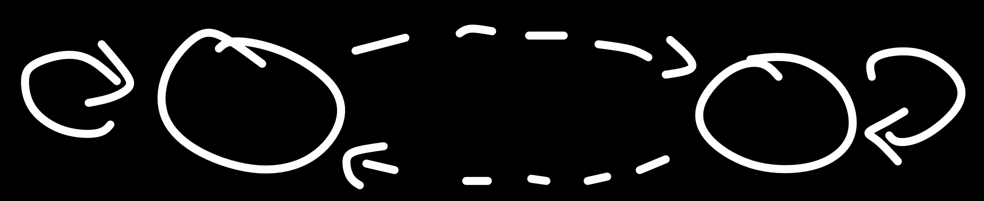
Coro $\mathbb{V}$ decidable $\iff$ $\mathbb{V}^{\ell}_{sync}$ decidable

Overview



Universe	$(\Sigma_\perp^2)^+$ $\begin{pmatrix} a & b & \perp \\ a & b & b \end{pmatrix}$, $\begin{pmatrix} a & \perp & c \\ \perp & b & a \end{pmatrix}$	$S_2\Sigma$ = well-formed $\begin{pmatrix} a & b & \perp \\ a & b & b \end{pmatrix}$
Algebras	Semigroups	Synchronous algebras
Finite rec.	Regular relations	
Algebraic char for rela° ?	✗	✓

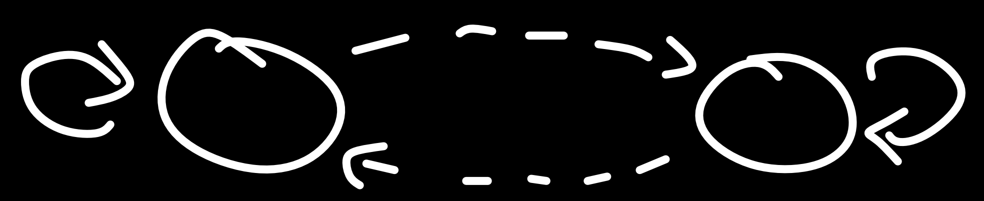
Beyond regular relations

<u>Graph</u> $\Omega =$ 

Monad of paths

Ω-path algebras

Beyond regular relations

Graph Ω = 

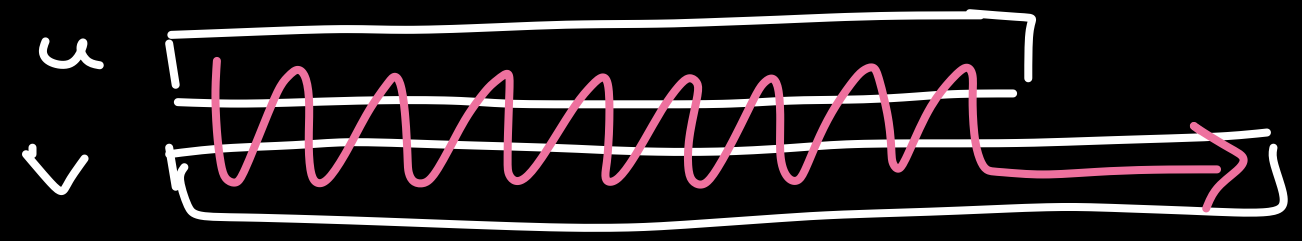
$\Downarrow$

Monad of paths

$\Downarrow$

Ω-path algebras

Constrained automata



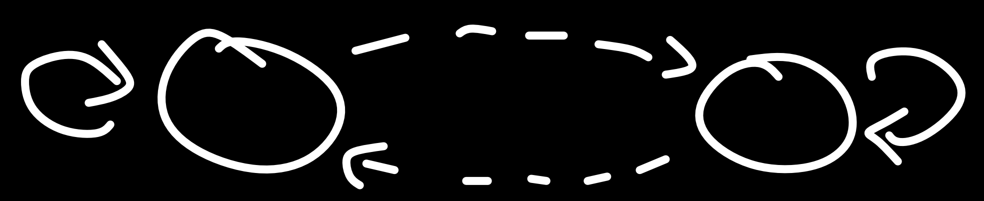
sync. auto

$(112)^*(1^* + 2^*)$ - auto

$(1+2)^*$ - auto = no constraint

= asynchronous automata

Beyond regular relations

Graph Ω = 

$\Downarrow$

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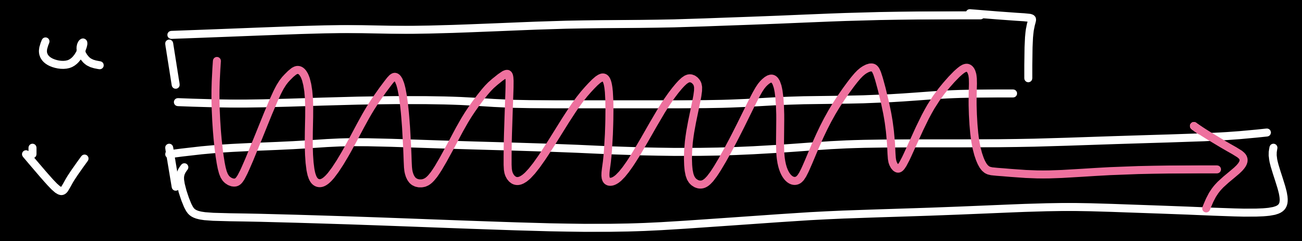
Ω-path algebras

Constrained automata



sync. auto

$(112)^*(1^*+2^*)$ - auto

$(1+2)^*$ - auto $=$ no constraint

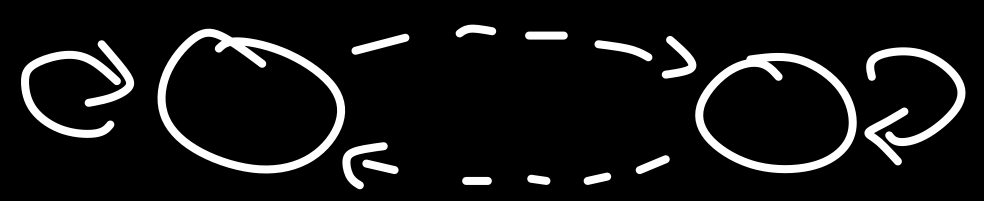
$=$ asynchronous automata

Open problem [Figueira, Libkin, STACS '14]

[Descotte, Figueira, Puppis, ICALP '18]

L_1-auto $\subseteq$ L_2-auto ?

Beyond regular relations

Graph Ω = 

$\Downarrow$

Monad of paths

$\Downarrow$

Ω-path algebras

Open problem [Figueira, Libkin, STACS '14]
[Descotte, Figueira, Puppis, ICALP '18]

L_1-auto $\subseteq$ L_2-auto ?

Constrained automata

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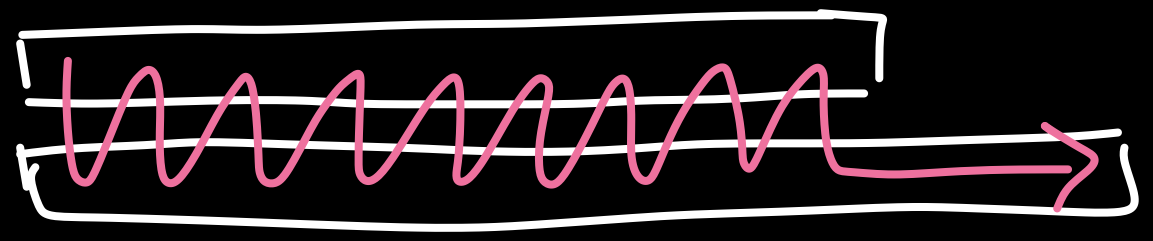
v

sync. auto

u
v

$(112)^*(1^*+2^*)$ - auto

$(1+2)^*$ - auto = no constraint
= asynchronous automata

Conj 1 L-auto $\cong$ Ω_L-algebras

Conj 2 L_1-auto $\subseteq$ L_2-auto
IFF
$\mathrm{Alg}(\Omega_{L_2}) \overset{T}{\rightleftarrows} \mathrm{Alg}(\Omega_{L_2})$