Pebble Relation Comonad and Pathwidth

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Resources and Co-resources Workshop

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Pebbling Comonad and Treewidth

Theorem ([ADW17])

The following are equivalent:

- $\mathcal{A}$ *has a tree-decomposition of width* $< k$
- $\mathcal{A}$ *has* k*-pebble forest cover*
- *There exists a coalgebra* $\alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A}$

Corollary ([ADW17])

$twd(\mathcal{A}) = k - 1$ *if and only if* k *is least index for which there exists a coalgebra* $\alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A}$

(Tree/Path)-Decomposition

Definition

A tree decomposition of $\mathcal{A}$ of width k is a triple $(T, \leq_T, \lambda : T \to \mathcal{P}A)$

- For $a \in A$, there is an $x \in T$ where $a \in \lambda(x)$
- If $a \in \lambda(x) \cap \lambda(x')$, then $a \in \lambda(y)$ for every y in $\text{path}(x, x')$
- If $a \frown a' \in \mathcal{A}$, then $\{a, a'\} \subseteq \lambda(x)$ from some $x \in T$
- $k = \max\{|\lambda(x)|\}_{x \in T} - 1$

If $\leq_T$ is well-ordered, then $(T, \leq_T, \lambda)$ is a path decomposition of $\mathcal{A}$ of width k.

k-Pebble (Linear) Forest Cover

Definition

A k-pebble forest cover of $\mathcal{A}$ is a tuple $(\{(T_i, \leq_i)\}, p : A \to [k])$ where $\{(T_i, \leq_i)\}$ is a family of disjoint trees.

- If $a \frown a' \in \mathcal{A}$, then there is a T_i such that $a, a' \in T_i$
- If $a \frown a' \in \mathcal{A}$ and $a \leq_i a'$, then for all $b \in (a, a]_{\leq_i}$, $p(b) \neq p(a)$

If every $\leq_i$ is a well-order, then $(\{(T_i, \leq_i)\}, p)$ is a k-pebble linear forest cover of $\mathcal{A}$.

Pebble Relation Comonad

Given a σ-structure $\mathcal{A}$ with universe A, define the set

$$\mathbb{PR}_k A := \{([(p_1, a_1), \ldots, (p_n, a_n)], i) \mid (p_j, a_j) \in \mathrm{k} \times A \text{ and } i \in \mathrm{n}\}$$

Let $\epsilon_A(s, i)$ be the i-th element of s and $\pi_A(s, i)$ be the i-th pebble of s. For $i < j$, let $s(i, j]$ denote the subsequence of s starting at $i + 1$ and ending at j. Otherwise, $s(i, j]$ is empty list. We can lift $\mathbb{PR}_k A$ to a σ-structure $\mathbb{PR}_k \mathcal{A}$:

$$\begin{aligned} R^{\mathbb{PR}_k \mathcal{A}}((s, i_1), (s, i_2)) \Leftrightarrow & \text{ let } i = \max(i_1, i_2), \\ & \text{then } \pi_A(s, i_j) \text{ does not appear in } s(i_j, i] \\ & \text{and } R^{\mathcal{A}}(\epsilon_A(s, i_1), \epsilon_A(s, i_2)) \end{aligned}$$

$(\mathbb{PR}_k, \epsilon, \delta)$ is a comonad over $\mathcal{R}(\sigma)$

Coalgebras over the Pebbling Comonad

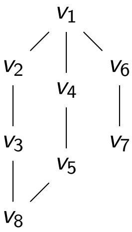

Figure: $\mathcal{A}$

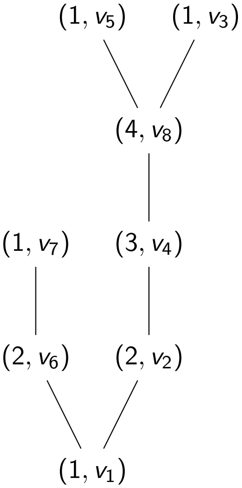

Figure: $\mathrm{im}(\alpha) \subset \mathbb{P}_4\mathcal{A}$

Coalgebras over the Pebble Relation Comonad

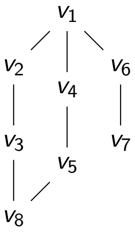

Figure: $\mathcal{A}$

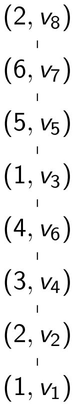

Figure: $\mathrm{im}(\alpha) \subset \mathbb{PR}_6\mathcal{A}$

Pebble Relation Comonad and Pathwidth

Theorem

The following are equivalent:

- $\mathcal{A}$ *has a* path-*decomposition of width* $< k$
- $\mathcal{A}$ *has k-pebble linear forest cover*
- *There exists a coalgebra* $\alpha : \mathcal{A} \to \mathbb{PR}_k\mathcal{A}$

Corollary

pwd$(\mathcal{A}) = k - 1$ *if and only if k is the least index for which there exists a coalgebra* $\alpha : \mathcal{A} \to \mathbb{PR}_k\mathcal{A}$

Path Decomposition $\Rightarrow$ Linear Forest Cover

Given a path decomposition $(T, \leq_T, \lambda)$ for $\mathcal{A}$ we can convert this to a linear forest cover:

- For each $x \in T$, we can define an injective function from $\tau_x : \lambda(x) \to [k]$ such that $\tau_x|_{(\lambda(x) \cap \lambda(x))} = \tau_{x'}|_{(\lambda(x) \cap \lambda(x'))}$.
- "Glue" the τ_x to obtain $p : A \to [k]$
- A new S_i for each connected component of A
- Let $x_a \in T$ least such that $a \in \lambda(x_a)$. $a \leq_i a'$ if $x_a <_T x'_a$ or $\tau_x(a) \leq \tau_x(a')$ if $x = x_a = x_{a'}$

Path Decomposition $\Leftarrow$ Linear Forest Cover

Given a k-pebble linear forest cover $(\{S_i, \leq_i)\}, p)$ for $\mathcal{A}$ we can convert this to a path decomposition:

- Let $(A, \leq_A)$ be our underlying path where $\leq_A$ is the ordered sum of the $(S_i, \leq_i)$
- Call a' an *active predecessor* of a if $a' \leq_i a$ and for all $b \in (a', a]$, $p(b) \neq p(a')$. Let $\lambda(a)$ be the set of active predecessor of a.

Linear Forest Cover $\Leftrightarrow$ $\mathbb{PR}_k$-Coalgebra

Given a k-pebble linear forest cover $(\{S_i, \leq_i\}, p)$ for $\mathcal{A}$ we can convert this to a $\alpha : \mathcal{A} \to \mathbb{PR}_k\mathcal{A}$

For S_i of the form

$$a_1 \leq_i \cdots \leq_i a_n$$

$t_i = [(p(a_1), a_1), \ldots, (p(a_n), a_n)]$.

$\alpha(a_j) = (t_i, j)$

Definition ([Dal05])

Consider the fragment of $M^k \subseteq \exists^+ L^k$ where conjunctions are of the form $\bigwedge \Psi$ for Ψ satisfying the conditions:

Every formula in Ψ with more than $k-1$ variables is quantifier-free.

At most one formula in Ψ containing quantifiers is not a sentence.

$$\phi_1(x, y) = E(x, y) \in M^3$$

$$\phi_{n+1}(x, y) = \exists z(E(x, z) \wedge \exists x(x = z \wedge \phi_n(x, y))) \in M^3$$

Theorem ([Dal05])

(1) Duplicator has a winning strategy in the k-Pebble Relation game from $\mathcal{A}$ to $\mathcal{B}$

(2) For every sentence $\phi \in M^k$, $\mathcal{A} \models \phi \Rightarrow \mathcal{B} \models \phi$

(3) For every σ-structure P *with pathwidth at most $k-1$,*

P $\to \mathcal{A} \Rightarrow$ P $\to \mathcal{B}$ *(denote "$\mathcal{A} \xrightarrow{pwd<k} \mathcal{B}$")*

Pebble Relation Game

The Pebble-Relation game from $\mathcal{A}$ to $\mathcal{B}$ is played as follows:

- Game begins with $I = \varnothing$ and $T = \hom(\varnothing, \mathcal{B})$
- For I' and T' of the previous move,
  - Spoiler shrinks the window $I \subseteq I'$,
    - Duplicator chooses restrictions of T' to I
  - Spoiler grows the window $I' \subseteq I$ (w/ $|I| \leq k$)
    - Duplicator responds with a set of homomorphisms T which are extensions of functions of some $S' \subseteq T'$ to I
- Spoiler wins if Duplicator can't successfully extend any of the homomorphisms

Duplicator has the advantage of non-determinism

Linking theorem

Theorem

Let $\mathbb{C}$ is a comonad. $f : \mathbb{C}A \to B$ if and only if for all coalgebras $\alpha : D \to \mathbb{C}D$,

$$D \to A \Rightarrow D \to B$$

Denote the condition on the RHS as $A \xrightarrow{\mathbb{C}} B$. Therefore, $\mathbb{C}A \to B \Leftrightarrow A \xrightarrow{\mathbb{C}} B$

Proof.

$\Rightarrow$ Suppose $h : D \to A$, then $f \circ \mathbb{C}h \circ \alpha : D \to B$
$\Leftarrow$ Choosing $\alpha = \delta_A : \mathbb{C}A \to \mathbb{C}\mathbb{C}A$ and the fact that $\epsilon_A : \mathbb{C}A \to A$ exists, then by the hypothesis $f : \mathbb{C}A \to B$ □

Corollary

There exists a morphism $f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B}$ *iff for all* $\phi \in M^k$ $\mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi$

Proof.

$$\begin{aligned} f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B} &\Leftrightarrow \mathcal{A} \xrightarrow{\mathbb{PR}_k} \mathcal{B} \text{ linking theorem} \\ &\Leftrightarrow \mathcal{A} \xrightarrow{pwd<k} \mathcal{B} \text{ characterization theorem} \\ &\Leftrightarrow \forall\phi \in M^k, \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi \text{ Dalmau's theorem} \end{aligned}$$

□

Where this is a going?

Back-and-forth equivalence? CoKleisli Isomorphisms to add counting quantifiers?

$\mathcal{B}$ has bounded treewidth duality, $\exists k \forall \mathcal{A}$, $\mathbb{P}_k \mathcal{A} \to \mathcal{B} \Leftrightarrow \mathcal{A} \to \mathcal{B}$

- $\mathcal{B}$ has bounded treewidth duality $\Rightarrow$ CSP($\mathcal{B}$) $\in$ PTIME
- Converse does not hold [FV98].

$\mathcal{B}$ has bounded pathwidth duality, $\exists k \forall \mathcal{A}$, $\mathbb{PR}_k \mathcal{A} \to \mathcal{B} \Leftrightarrow \mathcal{A} \to \mathcal{B}$

- $\mathcal{B}$ has bounded pathwidth duality $\Rightarrow$ CSP($\mathcal{B}$) $\in$ NL
- Converse is a open problem [Dal05].
- CSP($\mathcal{B}$) definable in Krom SNP $\Rightarrow$ CSP($\mathcal{B}$) $\in$ NL
- CSP($\mathcal{B}$) $\in$ NL $\Rightarrow$ CSP($\mathcal{B}$) definable in Krom SNP $+0+$ succ

Comonad capturing symmetric pathwidth duality? (Possibly rotation list comonad)

Samson Abramsky, Anuj Dawar, and Pengming Wang.
The pebbling comonad in finite model theory.
In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, June 2017.

Victor Dalmau.
Linear Datalog and Bounded Path Duality of Relational Structures.
Logical Methods in Computer Science, 1(1):5, April 2005.
arXiv: cs/0504027.

Tomás Feder and Moshe Y. Vardi.
The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory.
SIAM Journal on Computing, 28(1):57–104, January 1998.