Diagrammatic Algebra of First Order Logic

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Two starting points

- **Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories**
  - **an algebra of relations with the expressive power of regular logic**

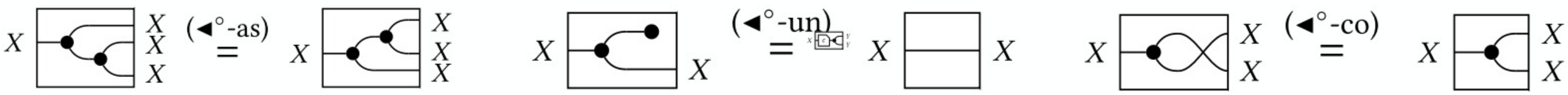
- Charles Peirce’s Calculus of Relations (1883)
  - featuring linear distributivity and linear adjoints

Towards cartesian bicategories i

- Lawvere in the 1960s realised the power of **cartesian categories**
  - **free cartesian categories** on a signature are the same as categories of terms and substitutions (**classical syntax**)
  - cartesian category induced by a (presentation of an) algebraic theory is a **presentation-independent** notion of algebraic theory in the universal algebraic sense
  - **functorial semantics**: models are cartesian functors to Set, homomorphisms are natural transformations

Aside - Fox's theorem

- A category is cartesian iff it is symmetric monoidal st every object is equipped with a cocommutative **comonoid structure**



- which is **natural**

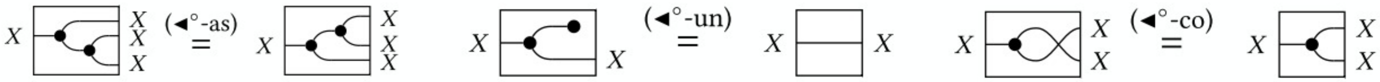
- and **coherent**

Towards cartesian bicategories ii

- But what if one wants to move to more expressive theories?
  - e.g. what if one wants models in **Rel**?
  - **Rel** = category with objects sets and arrows $X \to Y$ relations $R \subseteq X \times Y$
    - composition $x\ (R\ ;\ S)\ z$ iff $\exists y.\ xRy \wedge ySz$
    - identies are $x\ I\ y$ iff $x=y$
- Cartesian product is **still** important (n-ary relations can be seen as a relation of type $X^n \to 1$)
- But cartesian product is **not** the categorical product in **Rel**…
- Note though: it does make **Rel** a symmetric monoidal category and every homset is a poset

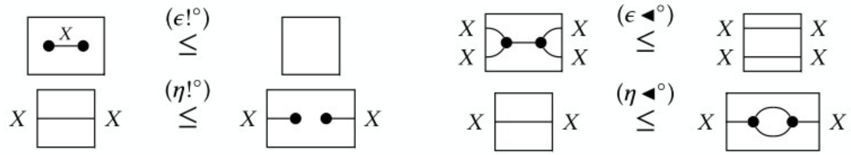
Cartesian bicategories

- every homset is a poset
- every object X is equipped with a cocommutative comonoid structure

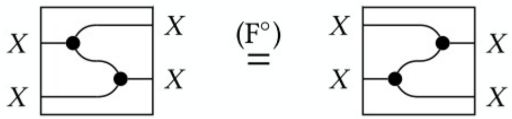


- but now the naturality is only weak

- and there is new structure!
    - the comonoid structure has **right adjoints**



    - and together they satisfy the **Frobenius equation**



Functorial semantics for relational theories

- A la Lawvere, once you know that the notion of cartesian bicategory replaces cartesian category
  - term syntax is given by string diagrams
  - models are functors of cartesian bicategories to **Rel**
  - homomorphisms are the canonical notion of natural transformation
- completeness (CSL 2018)
- This same general functorial semantics recipe is repeated for partial algebraic theories (PoPL 21) and coherent theories (PoPL 23)

Two starting points

- Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories
  - an algebra of relations with the expressive power of regular logic

- **Charles Peirce's Calculus of Relations (1883)**
  - **featuring linear distributivity and linear adjoints**

Aside: **Rel**'s weird cousin

- From now on let us call the usual category of relations **Rel**$^\circ$
- Lets meet its strange cousin, **Rel**$^\bullet$
  - objects are still sets and arrows are still relations
  - composition is $x\ (R\ ;\ S)\ z$ iff $\forall\ y.\ xRy \vee ySz$
  - identities are $x\ I\ y$ iff $x \neq y$
  - cartesian product on objects still makes it a symmetric monoidal category, and homsets are posets
- But it is a **cocartesian bicategory** (the inequalities go the other way!)

Peirce's calculus of relations (1883)

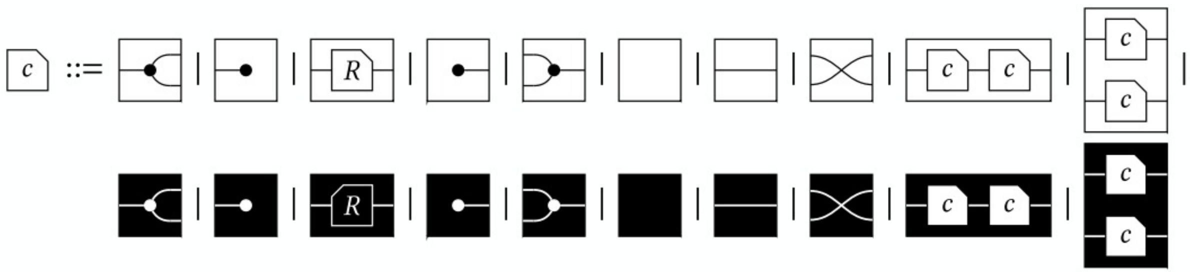
- Peirce liked the weird cousin

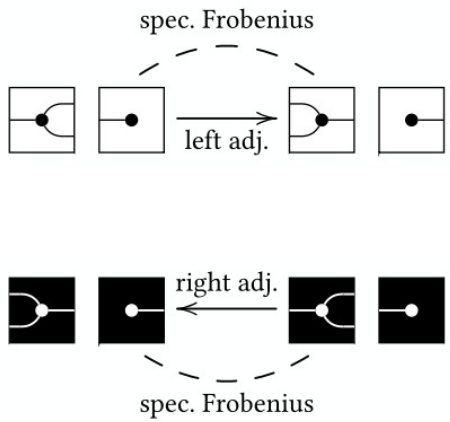
$$E \quad ::= \quad R \mid id^{\circ} \mid E \mathbin{\circ\!\!\!,} E \mid id^{\bullet} \mid E \mathbin{\bullet\!\!\!,} E \mid \bot \mid E \cup E \mid \top \mid E \cap E \mid E^{\dagger} \mid \overline{E}$$

- The calculus only deals with binary relations. Peirce did not like this and went on to work on **existential graphs** (19th century string diagrams)
- Later work on relational calculi (e.g. Tarski) discarded the "black" structure

Diagrams in Rel•

- Use black background/white strings to emphasise the “De Morgan” aspects

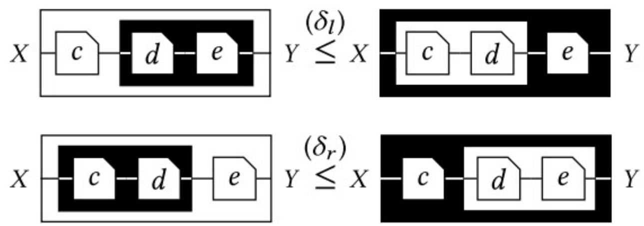




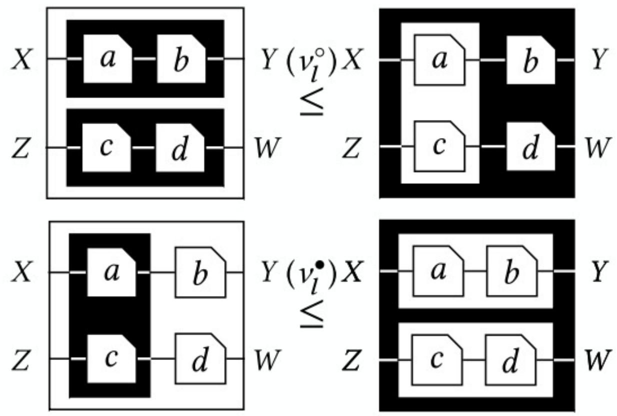
- but how to understand two compositions and two tensors together?

(symmetric monoidal) Linear bicategories

- obvious extension of Cockett, Koslowski, Seely 2000
- linear distributivity

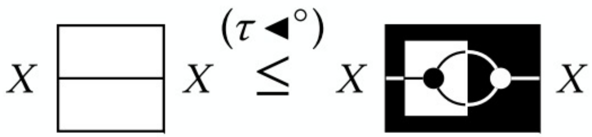


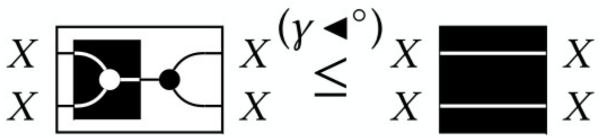
- and linear strengths for tensors
- + obvious laws for identities and symmetries



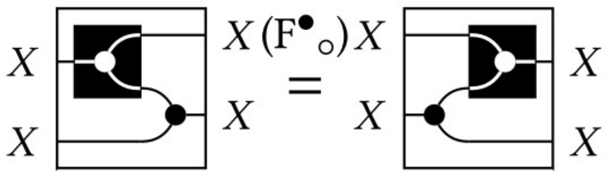
First order bicategories

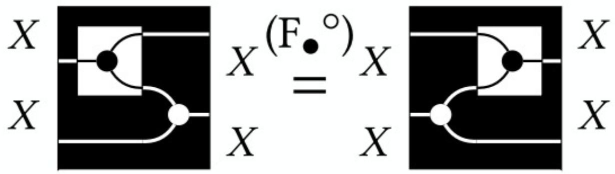
- The missing thing is to characterise how the two (co)cartesian structures interact:
- there are **linear adjunctions**
  - e.g.



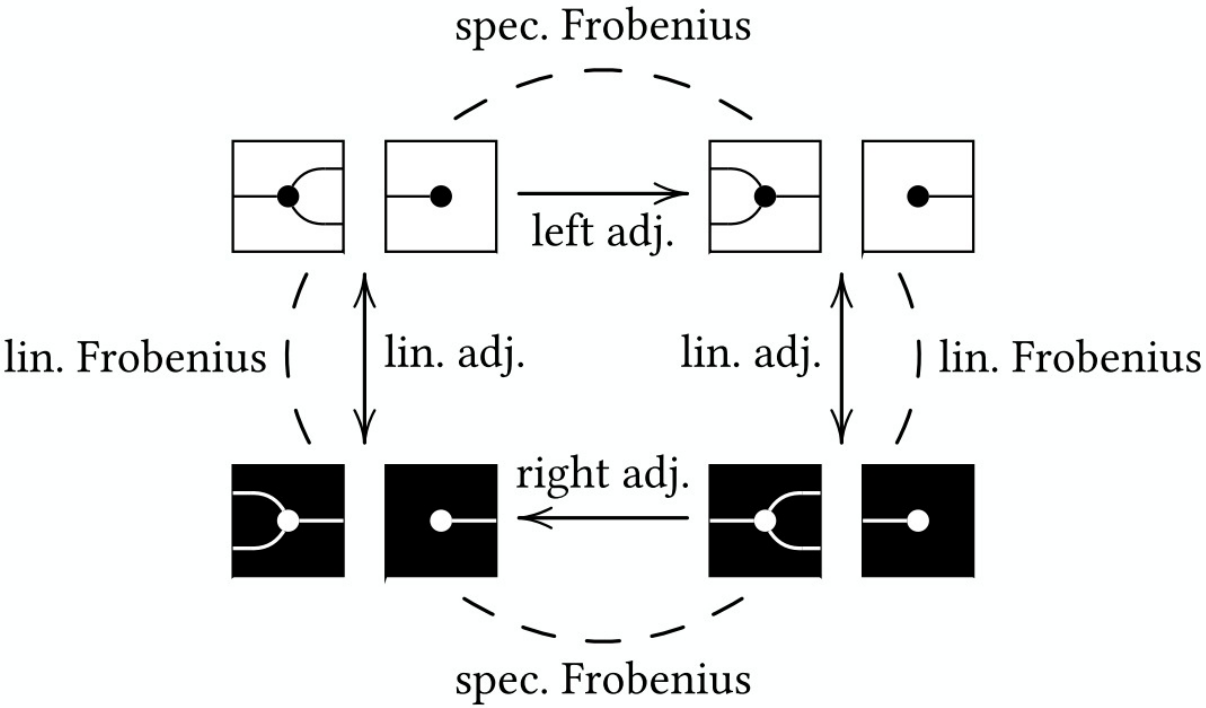


- … and “linear” Frobenius

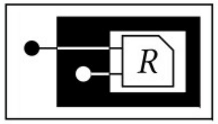




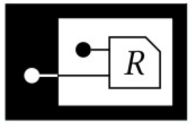
Summarising



Worked example



$\exists x.\, \forall y.\, R(x, y)$



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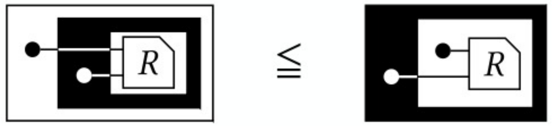

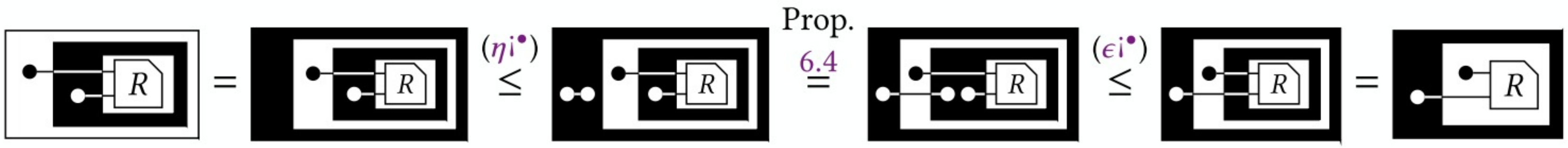

Highlights

- Gödel completeness by adapting Henkin's proof to the string diagrammatic language (more on this on the next slide)
- Functorial semantics for first order theories following the usual recipe
- No variables, no quantifiers
    - Easy and natural encodings of other variable free approaches (e.g. Quine predicate functor logic)

What's new, different?

- Diagrammatic syntax is closely related to Peirce's existential graphs
  - Although negation **is not** a primitive
    - it is a derived operation that operates on syntax
    - e.g. $\neg\neg\neg\phi$ is **syntactically equal** as a diagram to $\neg\phi$
- string diagrams let one to discover places where the traditional syntax has caused problems
  - trivial vs contradictory theories is a meaningful distinction
- trivial theories are propositional logic
  - our axiomatisation becomes Guglielmi's deep inference Calculus of Structures (SKSg)
- completeness theorem extends Gödel's to all theories