Categorical Structure in Theory of Arithmetic[a]

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[a] This talk is based on an eponymous paper by me.

An function $f : \mathbb{N}^k \to \mathbb{N}$ is *provably recursive* in $\mathbb{T}$ if:

- There is a Σ_1-formula $\varphi_f(\bar{x}, y)$ defines f;
- $\mathbb{T} \vdash \forall \bar{x} \exists_! y \, \varphi_f(\bar{x}, y)$.

Logicians have considered many different arithmetic systems:

- PA, $I\Sigma_n$, EA, S_2^n, ...

Theorem (⋆)

Provably recursive functions in $I\Sigma_1$ are exactly p.r. functions.

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Complexity and Arithmetic

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Bridge from Categorical Logic

Every theory $\mathbb{T}$ has a *syntactic category* $\mathcal{C}[\mathbb{T}]$ encoding it:

- Objects are formulas in $\mathbb{T}$;
- Morphisms $\theta : \varphi(\bar{x}) \to \psi(\bar{y})$ are provably functional formulas,

$$\mathbb{T} \vdash \forall \overline{xy}(\theta(\bar{x}, \bar{y}) \to \varphi(\bar{x}) \wedge \psi(\bar{y})) \wedge \forall \bar{x}(\varphi(\bar{x}) \to \exists_! \bar{y} \theta(\bar{x}, \bar{y}))$$

Proposition

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$$\mathrm{Fun}^*(\mathcal{C}[\mathbb{T}], \mathbf{Set}) \simeq \mathrm{Mod}(\mathbb{T}).$$

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Strategy i

Look at a *coherent* theory of arithmetic $\mathbb{T}$:

- Formulas only contain $\top, \bot, \wedge, \vee, \exists$, with sequents $\varphi \vdash_{\bar{x}} \psi$;
- Usual peano axioms plus the following induction rule:

$$\frac{\varphi(\bar{x}) \vdash_{\bar{x}} \psi(\bar{x}, 0) \qquad \varphi(\bar{x}) \wedge \psi(\bar{x}, y) \vdash_{\bar{x},y} \psi(\bar{x}, \mathsf{s}y)}{\varphi(\bar{x}) \vdash_{\bar{x},y} \psi(\bar{x}, y)}$$

You may think of $\mathbb{T}$ as the Π_2-fragment of $I\Sigma_1$.

In particular, use $[n]$ to denote $\bigwedge_{1 \leq i \leq n} x_i = x_i$:

Observation

Morphisms from $[n]$ to $[1]$ in $\mathcal{C}[\mathbb{T}]$ are exactly provably recursive formulas in $\mathbb{T}$ ($I\Sigma_1$).

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Strategy ii

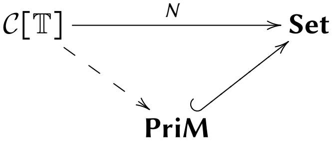
According to categorical logic, the standard model $\mathbb{N}$ induces:

$$\mathcal{C}[\mathbb{T}] \xrightarrow{\quad N \quad} \mathbf{Set}$$

And N maps every $\theta : [n] \to [1]$ to a provably recursive $\mathbb{N}^n \to \mathbb{N}$.

Strategy ii

The question now is whether we have a factorisation:



Here **PriM** morally is a category with:

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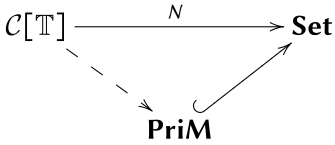

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Conclusion: ($\star$) is equivalent to the above factorisation.

Strategy iii

Technical Goal

To prove $\mathcal{C}[\mathbb{T}]$ is *initial* among some class of categories that include **PriM** and **Set**.

The relevant class is *coherent categories equipped with a PNO*:

- *Coherent*: categories that can interpret coherent logic;
- *PNO*: parametrised natural number object.

Examples:

- **Set**, **PriM**, any topos with a NNO, ...

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Proof

Aim: Prove $\mathcal{C}[\mathbb{T}]$ is initial in this class:

- The syntactic category of any coherent theory is coherent.
- Show $[1]$ is a PNO in $\mathcal{C}[\mathbb{T}]$.

After that, for any $(\mathcal{E}, N)$, there is an essentially unique coherent functor that maps $[1]$ to N, because $\mathcal{C}[\mathbb{T}]$ is "generated" by $[1]$.

Slogan

$(\star)$ is true precisely by the following *structural* reasons:

- Π_2-fragment of $I\Sigma_1$ presents the *initial* coh. cat. with a PNO.
- **PriM** *is* a coherent category with a PNO.

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Perspective

- *Initiality* is at the center of understanding $\mathbb{T}$ ($I\Sigma_1$).
  - Glueing argument (Tait computability) proves Σ_1-completeness.
  - Initiality also implies other constructive features of $\mathbb{T}$.
- Other arithmetic theories, other categorical treatment of complexity classes:
  - PA presents the *initial* Boolean category with a PNO.
  - J. Otto has a categorical treatment of PTIME.

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