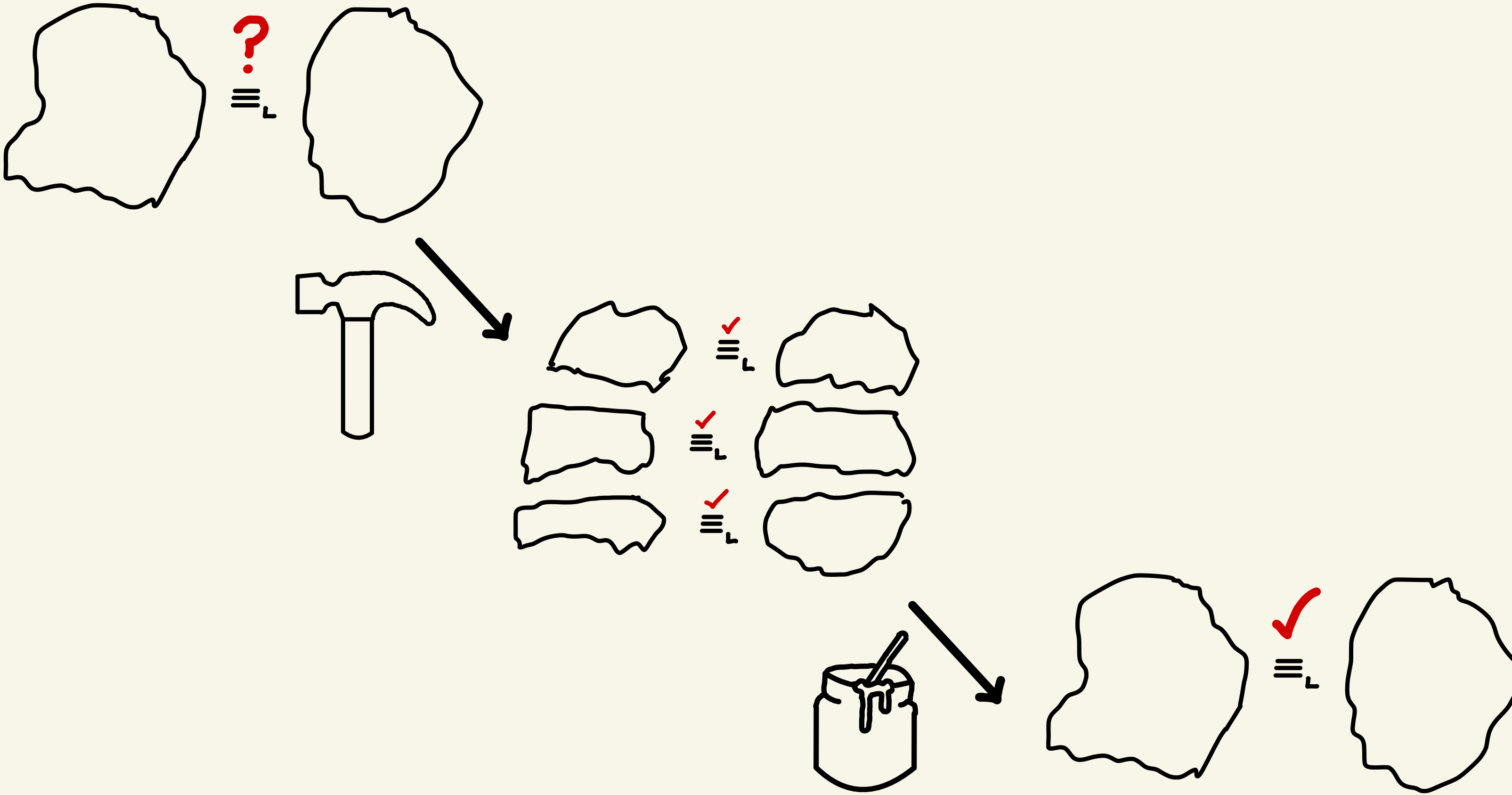
A categorical account of composition methods in logic

(Joint with Tomáš Jakl and Nihil Shah)

The composition method (a.k.a. the decomposition method)



Feferman–Vaught–Mostowski Theorems

Typical Formulation

If we have model equivalences:

$$A_1 \equiv_{\mathcal{L}_1} B_1, \dots, A_n \equiv_{\mathcal{L}_n} B_n$$

Then

$$Op(A_1, \dots, A_n) \equiv_{\mathcal{L}} Op(B_1, \dots, B_n)$$

Examples

1) If $A_1 \equiv_{FO} B_1$ and $A_2 \equiv_{FO} B_2$ then:
   - $A_1 \times A_2 \equiv_{FO} B_1 \times B_2$
   - $A_1 + A_2 \equiv_{FO} B_1 + B_2$
2) The same for MSO and coproducts, but <u>not</u> products
3) (A non-example) If $A_1 \equiv_{ML} B_1$ and $A_2 \equiv_{ML} B_2$ then not necessarily $A_1 + A_2 \equiv_{ML} B_1 + B_2$

Game Comonad Equivalences

For a game comonad $\mathbb{C}$, we consider:

1) $A \rightleftarrows B$ in $R(\sigma)_{\mathbb{C}}$, denoted $A \equiv_{\exists^+\mathbb{C}} B$ and $A \Rrightarrow_{\exists^+\mathbb{C}} B$

2) $A \cong B$ in $R(\sigma)_{\mathbb{C}}$, denoted $A \equiv_{\#\mathbb{C}} B$

3) There is a span of open pathwise-embeddings in $R(\sigma)^{\mathbb{C}}$

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ FA & & FB \end{tikzcd}$$

, denoted $A \equiv_{\mathbb{C}} B$

Example

For the Ehrenfeucht-Fraisse comonad $\mathbb{E}_k$

1) Corresponds to equivalence in the $\exists^+$ fragment of FO
2) Corresponds to equivalence in FO extended with counting
3) Corresponds to FO equivalence

All up to quantifier depth k.

Game Comonads and FVM Theorems

In the setting of game comonads, there are three natural classes of FVM theorem to consider:

1) $A_1 \equiv_{\exists^+ \mathbb{C}_1} B_1, \ldots, A_n \equiv_{\exists^+ \mathbb{C}_n} B_n$ implies $Op(A_1, \ldots, A_n) \equiv_{\exists^+ \mathbb{D}} Op(B_1, \ldots, B_n)$

2) $A_1 \equiv_{\# \mathbb{C}_1} B_1, \ldots, A_n \equiv_{\# \mathbb{C}_n} B$ implies $Op(A_1, \ldots, A_n) \equiv_{\# \mathbb{D}} Op(B_1, \ldots, B_n)$

3) $A_1 \equiv_{\mathbb{C}_1} B_1, \ldots, A_n \equiv_{\mathbb{C}_n} B_n$ implies $Op(A_1, \ldots, A_n) \equiv_{\mathbb{D}} Op(B_1, \ldots, B_n)$

Dealing with $\Rrightarrow_{\exists^+}$

Consider binary $\mathbf{H}$. If $A_1 \equiv_{\exists^+ \mathbb{C}_1} B_1$, $A_2 \equiv_{\exists^+ \mathbb{C}_2} B_2$, we then have morphisms

$$\mathbb{C}_1 A_1 \xrightarrow{t_1} B_1 \quad \text{and} \quad \mathbb{C}_2 A_2 \xrightarrow{t_2} B_2$$

<u>Assuming $\mathbf{H}$ is functorial</u>, we can form

$$\mathbf{H}(\mathbb{C}_1 A_1, \mathbb{C}_2 A_2) \xrightarrow{\mathbf{H}(t_1, t_2)} \mathbf{H}(B_1, B_2)$$

Dealing with $\Rrightarrow_{\exists^+}$ (Continued)

Assuming a family of morphisms, we can form the composite.

$$\underbrace{\mathbb{D}H(A_1, A_2) \xrightarrow{\kappa_{A_1,A_2}} H(\mathbb{C}_1 A_1, \mathbb{C}_2 A_2)} \xrightarrow{H(f_1, f_2)} H(B_1, B_2)$$

and so

$$H(A_1, A_2) \Rrightarrow_{\exists^+ \mathbb{D}} H(B_1, B_2)$$

The General $\Rrightarrow_{\exists^+}$ and $\equiv_{\exists^+}$ FVM Theorem

Theorem III.2. *Let $\mathbb{C}_1, \ldots, \mathbb{C}_n$ and $\mathbb{D}$ be comonads on categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$, and $\mathcal{D}$ respectively, and*

$$H \colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

a functor. If for every $A_1 \in \mathrm{obj}(\mathcal{C}_1)$, …, $A_n \in \mathrm{obj}(\mathcal{C}_n)$ there exists a morphism

$$\mathbb{D}(H(A_1, \ldots, A_n)) \xrightarrow{\kappa_{A_1, \ldots, A_n}} H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n)) \quad (4)$$

in $\mathcal{D}$, then

$$A_1 \Rrightarrow_{\exists^+ \mathbb{C}_1} B_1, \ \ldots, \ A_n \Rrightarrow_{\exists^+ \mathbb{C}_n} B_n$$

implies

$$H(A_1, \ldots, A_n) \Rrightarrow_{\exists^+ \mathbb{D}} H(B_1, \ldots, B_n)$$

The same result holds when replacing $\Rrightarrow_{\exists^+ \mathbb{C}}$ with $\equiv_{\exists^+ \mathbb{C}}$.

Dealing with uncoy $\equiv_{\#}$

Assuming $A \equiv_{\#\mathbb{C}} B$, there are morphisms:

$$f : \mathbb{C}A \to B \quad \text{and} \quad g \; \mathbb{C}B \longrightarrow A$$

which are mutually inverse in the Kleisli category of $\mathbb{C}$.

Continuing from the assumptions used for $\equiv_{\exists +}$, we would like the composites:

$$\mathbb{D}\,\mathbb{H}A \xrightarrow{\kappa_A} \mathbb{H}\,\mathbb{C}A \xrightarrow{\mathbb{H}(f)} \mathbb{H}B \quad \text{and} \quad \mathbb{D}\,\mathbb{H}B \xrightarrow{\kappa_B} \mathbb{H}\,\mathbb{C}B \xrightarrow{\mathbb{H}(g)} \mathbb{H}A$$

are mutually inverse in the Kleisli category of $\mathbb{D}$.

Dealing with uncoy $\equiv_{\#}$

$$
\begin{aligned}
& (H(g) \cdot \kappa_B) \bullet (H(f) \cdot \kappa_A) \\
= \langle & \text{ definition } \rangle \\
& H(g) \cdot \kappa_B \cdot (H(f) \cdot \kappa_A)^* \\
= \langle & \text{ ? } \rangle \\
& H(g) \cdot H(f^*) \cdot \kappa_A \\
= \langle & \text{ functoriality } \rangle \\
& H(g \cdot f^*) \cdot \kappa_A \\
= \langle & \text{ definition } \rangle \\
& H(g \bullet f) \cdot \kappa_A \\
= \langle & \text{ assumption } \rangle \\
& \varepsilon_A \cdot \kappa_A \\
= \langle & \text{ ? } \rangle \\
& \varepsilon_{H(A)}
\end{aligned}
$$

Suggests two axioms:

1) $\kappa_B \cdot (H(f) \cdot \kappa_A)^* = H(f^*) \cdot \kappa_A$

2) $\varepsilon_A \cdot \kappa_A = \varepsilon_{H(A)}$

Dealing with unary $\equiv_\#$, Kleisli Laws

The axioms on the previous slide are equivalent to requiring

$$\mathbb{D} H(A) \xrightarrow{\kappa_A} H(\mathbb{C} A)$$

constitute a kleisli law. These correspond to liftings:

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{C}} \arrow[r, "\hat{H}"] & \mathcal{D}_{\mathbb{D}} \\
\mathcal{C} \arrow[u] \arrow[r, "H"'] & \mathcal{D} \arrow[u]
\end{tikzcd}$$

Dealing with $\equiv_{\#}$, the general case

n-ary Kleisli laws:

$$\mathbb{D} \circ H \xRightarrow{\kappa} H \circ \prod_i \mathbb{C}_i$$

bijectively correspond to liftings:

$$\begin{tikzcd}
\prod_i {C_i}_{\mathbb{C}_i} \arrow[r, "\hat{H}"] & D_{\mathbb{D}} \\
\prod_i C_i \arrow[u] \arrow[r, "H"'] & D \arrow[u]
\end{tikzcd}$$

The general $\equiv_{\#}$ FVM Theorem

Theorem IV.4. *Let $\mathbb{C}_1, \ldots, \mathbb{C}_n$ and $\mathbb{D}$ be comonads on categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and $\mathcal{D}$, respectively, and $H \colon \prod_i \mathcal{C}_i \to \mathcal{D}$ a functor. If there exists a Kleisli law of type*

$$\mathbb{D} \circ H \Rightarrow H \circ \prod_i \mathbb{C}_i$$

then

$$A_1 \equiv_{\#\mathbb{C}_1} B_1 \;\; \ldots, \;\; A_n \equiv_{\#\mathbb{C}_n} B_n$$

implies

$$H(A_1, \ldots, A_n) \;\equiv_{\#\mathbb{D}}\; H(B_1, \ldots, B_n).$$

FO_k Equivalence and Coproducts

Given spans:

$$\begin{tikzcd} & R_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\ FA_1 & & FB_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & R_2 \arrow[dl, "l_2"'] \arrow[dr, "r_2"] & \\ FA_2 & & FB_2 \end{tikzcd}$$

we would like to construct a span

$$\begin{tikzcd} & R \arrow[dl, "l"'] \arrow[dr, "r"] & \\ F(A_1+A_2) & & F(B_1+B_2) \end{tikzcd}$$

~~SCOPE~~

Focus: Spans of the right shape, ignoring open pathwise embedding issues

Plan A - Use Coproducts

$R(\sigma)^{\mathbb{E}k}$ has coproducts, so we can form the span:

$$\begin{tikzcd} & R_1 + R_2 \arrow[dl, "\ell_1+\ell_2"'] \arrow[dr, "r_1+r_2"] & \\ FA_1 + FA_2 & & FB_1 + FB_2 \end{tikzcd}$$

Wanted $F(A_1+A_2)$

Wanted $F(B_1+B_2)$

The feet are the wrong form - we seem to be stuck.

Note

There is a canonical map $F(A)+F(B) \longrightarrow F(A+B)$, but this will not in general be an open pathwise embedding. If our example was for products, the corresponding canonical map would point in the wrong direction.

Plan B - Direct Construction

We explicitly construct an 'interleaving' bifunctor $(-) \wr (-)$
This yields a span:

$$
\begin{tikzcd}
& R_1 \wr R_2 \arrow[dl, "l_1 \wr l_2"'] \arrow[dr, "r_1 \wr r_2"] & \\
FA_1 \wr FA_2 \arrow[d, phantom, "\cong" rotate=90] & & FB_1 \wr FB_2 \arrow[d, phantom, "\cong" rotate=90] \\
F(A_1 + A_2) & & F(B_1 + B_2)
\end{tikzcd}
$$

Questions

1) Where did this come from?
2) What is the general pattern and its scope?

Bilinear Maps

Recall for vector spaces X, Y and Z that a function

$$h : X \times Y \longrightarrow Z$$

is <u>bilinear</u> if

1) $\forall x \in X \quad h(x, -)$ is linear.
2) $\forall y \in Y \quad h(-, y)$ is linear.

Also recall there is a tensor product of vector spaces which is universal in that:

$$\frac{\text{linear maps } X \otimes Y \longrightarrow Z}{\text{bilinear maps } X \times Y \longrightarrow Z}$$

Question: Where does this structure come from abstractly?

Vector Spaces as Eilenberg-Moore Algebras

Let S be a semiring. There is a Set monad $\mathbb{V}_S$ with:

- $\mathbb{V}_S(A)$ (finitely supported) formal sums of the form:
$$\sum_i s_i a_i$$
- Unit $\quad \eta(a) = a \quad$ (the trivial sum)
- Multiplication:

$$\sum_i s_i \sum_j s_j a_{ij} \overset{\mu}{\mapsto} \sum_i \sum_j s_i s_j a_{ij}$$

Examples

- For arbitrary S $\;\mathrm{Set}^{\mathbb{V}_S}$ is the category of S-semimodules
- For a ring $\;\mathrm{Set}^{\mathbb{V}_R}$ is the category of R-modules
- For a field F $\;\mathrm{Set}^{\mathbb{V}_F}$ is the category of F vector spaces
- $\mathrm{Set}^{\mathbb{V}_{\mathbb{N}}}$ is the category of Abelian monoids
- $\mathrm{Set}^{\mathbb{V}_{\mathbb{Z}}}$ is the category of Abelian groups

Special Classes of Monads

Definitions

1) A <u>symmetric monoidal category</u> (SMC) is a category $\mathcal{V}$ with:
   - A unit object I
   - A bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$
   - Natural isomorphisms:

     $$I \otimes A \cong A \qquad A \otimes I \cong A \qquad (A \otimes B) \otimes C \cong A \otimes (B \otimes C) \qquad A \otimes B \cong B \otimes A$$

   Subject to several coherence axioms

2) A <u>strength</u> for a functor $T : \mathcal{V} \longrightarrow \mathcal{V}$ on an SMC is a n.t.

$$st : A \otimes T(B) \longrightarrow T(A \otimes B)$$

   Subject to coherence axioms w.r.t. the SMC structure.
   A <u>strong functor</u> is a functor with a strength

3) A <u>strong</u> monad is a monad $(\mathbb{T}, \eta, \mu)$ such that st satisfies additional coherence axioms w.r.t. η and μ.

Set Monads are all strong

Canonical strength for $(\mathrm{Set}, \times, 1)$

For Set monad $\mathbb{T}$ define:

$$\begin{aligned} st : A \times \mathbb{T}B &\longrightarrow \mathbb{T}(A \times B) \\ (a, t) &\longmapsto \mathbb{T}(\lambda y.(a, y))(t) \end{aligned}$$

Examples

- List monad

$$(a, [b_1, \ldots, b_n]) \mapsto [(a, b_1), \ldots, (a, b_n)]$$

- For $\mathbb{V}^S$

$$(a, \sum_i s_i b_i) \mapsto \sum_i s_i (a, b_i)$$

Commutative Monads

We can define a <u>dual strength</u> st' as:

$$st' := \mathbb{T}(A) \otimes B \xrightarrow{\cong} B \otimes \mathbb{T}(A) \xrightarrow{st} \mathbb{T}(B \otimes A) \xrightarrow{\mathbb{T}(\cong)} \mathbb{T}(A \otimes B)$$

We can then define two double strength maps:

$$dst := \mathbb{T}A \otimes \mathbb{T}B \xrightarrow{st} \mathbb{T}(\mathbb{T}A \otimes B) \xrightarrow{\mathbb{T}st'} \mathbb{T}^2(A \otimes B) \xrightarrow{\mu} \mathbb{T}(A \otimes B)$$

$$dst' := \mathbb{T}A \otimes \mathbb{T}B \xrightarrow{st'} \mathbb{T}(A \otimes \mathbb{T}B) \xrightarrow{\mathbb{T}st} \mathbb{T}^2(A \otimes B) \xrightarrow{\mu} \mathbb{T}(A \otimes B)$$

A monad is <u>commutative</u> if $dst = dst'$

Commutativity for Set Monads

List

$$dst([a_1,\dots,a_n],[b_1,\dots,b_n]) := [(a_1,b_1),\dots,(a_n,b_1),\dots,(a_1,b_m),\dots,(a_n,b_m)]$$

$$dst'([a_1,\dots,a_n],[b_1,\dots,b_m]) := [(a_1,b_1),\dots,(a_1,b_m),\dots,(a_n,b_1),\dots,(a_n,b_m)]$$

So list is <u>not</u> commutative.

$\mathbb{V}^S$

$$dst\Big(\sum_i s_i a_i, \sum_j r_j b_j\Big) := \sum_{ij} r_j s_i\,(a_i,b_j)$$

$$dst'\Big(\sum_i s_i a_i, \sum_j r_j b_j\Big) := \sum_{ij} s_i r_j\,(a_i,b_j)$$

So $\mathbb{V}^S$ is commutative iff S has a commutative multiplication

Algebraic Intuition

A Set monad presented by (Σ, E) is commutative iff all the operations in Σ are homomorphisms w.r.t. each other. <u>This is unrelated to commutativity in the sense</u> $a+b=b+a$.

Bilinearity Abstractly (Finally!)

Bimorphisms

For a commutative monad $\mathbb{T}$, and algebras (A,α), (B,β), (C,γ) we say that $h : A \otimes B \longrightarrow C$ is <u>bilinear</u> or a <u>bimorphism</u> if the following diagram commutes:

$$\begin{tikzcd}
\mathbb{T}A \otimes \mathbb{T}B \arrow[r, "dst"] \arrow[d, "\alpha \otimes \beta"'] & \mathbb{T}(A \otimes B) \arrow[r, "\mathbb{T}(h)"] & \mathbb{T}C \arrow[d, "\gamma"] \\
A \otimes B \arrow[rr, "h"'] & & C
\end{tikzcd}$$

Examples

- For vector spaces or (semi)modules over a commutative (semi)ring this is the usual notion of bilinearity.
- For Abelian monoids or groups this is the usual notion of bimorphism.

Some standard results

Classical Results of Kock (See also Jacobs, Seal)

If T is a commutative monad

1) The SMC structure lifts to a structure $\otimes_T$ on $\mathcal{V}_T$ such that:

$$A \otimes B = A \otimes_T B \quad \text{(on objects)}$$

2) If $\mathcal{V}^T$ has coequalizers of reflexive pairs the SMC structure lifts to $\otimes^T$ on $\mathcal{V}^T$ such that

    i) $F(A) \otimes^T F(B) \cong F(A \otimes B)$

    ii) The tensor is universal in that:

$$\begin{array}{c} \text{bimorphism } (\alpha, \beta) \longrightarrow \gamma \\ \hline\hline \text{morphisms } \alpha \otimes^T \beta \longrightarrow \gamma \end{array}$$

Back to logic

Dualising if we have a binary operation $\otimes$ such that:

1) $\otimes$ induces an SMC structure on $R(\sigma)$
2) We can identify a strength for our comonad of interest $\mathbb{C}_k$
3) Our comonad is commutative
4) $R(\sigma)^{\mathbb{C}_k}$ has equalizers of reflexive pairs

Then:

1) In $R(\sigma)_{\mathbb{C}_k}$

$$(f_1 : A_1 \longrightarrow B_1 \,,\, f_2 : A_2 \longrightarrow B_2) \mapsto A_1 \otimes A_2 = A_1 \otimes_{\mathbb{C}_k} A_2 \xrightarrow{f_1 \otimes_{\mathbb{C}_k} f_2} B_1 \otimes_{\mathbb{C}_k} B_2 = B_1 \otimes B_2$$

$\otimes$ preserves $\rightleftarrows$ and $\cong$ in $R(\sigma)_{\mathbb{C}_k}$

2) In $R(\sigma)^{\mathbb{C}_k}$

$$\left(\begin{tikzcd} & R_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\ FA_1 & & FB_1 \end{tikzcd} , \begin{tikzcd} & R_2 \arrow[dl, "l_2"'] \arrow[dr, "r_2"] & \\ FA_2 & & FB_2 \end{tikzcd} \right) \mapsto \begin{tikzcd} & R_1 \otimes^{\mathbb{C}_k} R_2 \arrow[dl] \arrow[dr] & \\ FA_1 \otimes^{\mathbb{C}_k} FA_2 \arrow[d, phantom, "\cong" rotate=90] & & FB_1 \otimes^{\mathbb{C}_k} FB_2 \arrow[d, phantom, "\cong" rotate=90] \\ F(A_1 \otimes A_2) & & F(B_1 \otimes B_2) \end{tikzcd}$$

We can construct spans of the right shape for two sided equivalence

Worst Talk Ever!

That was somewhat underwhelming

- We need to identify and verify monoidal structure
- We need to identify and verify a strength such that our comonad is commutative
- We need to verify a technical condition on $R(\sigma)^{\mathbb{C}_k}$

Even then:

- We can only deal with binary operations
- We can only work with a single base category (signature)
- We can only work with a single comonad (logic)

Questions

1) How much of this stuff do we really need?
2) What role do the various assumptions really play?
3) How far can we generalize?

Looking again at bimorphisms

For a functor $H : \mathcal{C} \longrightarrow \mathcal{D}$, monads $\mathbb{S} : \mathcal{C} \longrightarrow \mathcal{C}$ and $\mathbb{T} : \mathcal{D} \longrightarrow \mathcal{D}$, $\mathbb{S}$-algebra (A, α) and $\mathbb{T}$-algebra (B, β), and $\lambda : H\mathbb{S}A \longrightarrow \mathbb{T}HA$, $h : H(A) \longrightarrow B$ is a <u>bimorphism</u> if:

$$\begin{tikzcd}
H\mathbb{S}A \arrow[r, "\lambda"] \arrow[d, "H\alpha"'] & \mathbb{T}HA \arrow[r, "\mathbb{T}h"] & \mathbb{T}B \arrow[d, "\gamma"] \\
HA \arrow[rr, "h"'] & & B
\end{tikzcd}$$

Note
We make no assumptions on λ such as naturality at this point.

Note
Despite appearing unary, this really does generalize the previous condition, as we may consider product monads on product categories

Generalizing the classical results

1) If $\mathcal{D}^{\mathbb{T}}$ has coequalizers of reflexive pairs there exists an algebra $\hat{H}(\alpha)$ such that we have bijection:

$$\frac{\text{Bimorphisms } \alpha \longrightarrow \beta}{\text{Morphisms } \hat{H}\alpha \longrightarrow \beta}$$

2) If furthermore $\lambda : H\$A \longrightarrow \mathbb{T}HA$ is natural in A, then $\hat{H}$ extends to a functor $\mathcal{C}^{\$} \longrightarrow \mathcal{D}^{\mathbb{T}}$.

3) If also λ satisfies:

$$\begin{tikzcd} H\$A \arrow[rr, "\lambda"] & & \mathbb{T}HA \\ & HA \arrow[ul, "H\eta"] \arrow[ur, "\eta"'] \arrow[u, phantom, "1"] & \end{tikzcd}
\qquad
\begin{tikzcd} H\$^2A \arrow[r, "\lambda"] \arrow[d, "H\mu"'] \arrow[drr, phantom, "2"] & \mathbb{T}H\$A \arrow[r, "\mathbb{T}\lambda"] & \mathbb{T}^2HA \arrow[d, "\mu"] \\ H\$A \arrow[rr, "\lambda"'] & & \mathbb{T}HA \end{tikzcd}$$

Then $\hat{H}F(A) \cong FH(A)$.

Back to logic again

We now have the following for operation H

1) If H is functorial and there is a Kleisli-law $\lambda : H\mathbb{C}_k \Longrightarrow \mathbb{D}_\ell H$ then
   i) H preserves $\rightleftarrows$ equivalence
   ii) H preserves Kleisli isomorphism equivalence
2) If $\mathcal{D}^{\mathbb{D}_\ell}$ has equalizers of reflexive pairs then H preserves spans of the required shape.

Note

These results cover n-ary operations involving potentially different base categories and comonads.

The main theorem

Up to introduction of a suitable factorisation system, some assumptions about paths, and one remaining condition (S2') relating all three:

Theorem V.13. *Let $\mathbb{C}_1, \ldots, \mathbb{C}_n$ and $\mathbb{D}$ be comonads on categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and $\mathcal{D}$, respectively, and $H \colon \prod_i \mathcal{C}_i \to \mathcal{D}$ a functor which preserves embeddings. If there exists a Kleisli law of type*

$$\mathbb{D} \circ H \Rightarrow H \circ \prod_i \mathbb{C}_i$$

satisfying (S2'), then

$$A_1 \equiv_{\mathbb{C}_1} B_1, \ \ldots, \ A_n \equiv_{\mathbb{C}_n} B_n$$

implies

$$H(A_1, \ldots, A_n) \equiv_{\mathbb{D}} H(B_1, \ldots, B_n).$$

Example application

Theorem VI.4. *If $\mathbb{C}$ preserves embeddings and if for any surjective morphism of coalgebras $(A, \alpha) \to (B, \beta)$ in $\mathsf{EM}(\mathbb{C})$, if (A, α) is a path then so is (B, β), then*

$$\begin{aligned} & A_1 \equiv_{\mathbb{C}} A_2 \ \textit{ and } \ B_1 \equiv_{\mathbb{C}} B_2 \\ & \textit{implies} \quad A_1 \times A_2 \equiv_{\mathbb{C}} B_1 \times B_2 \end{aligned}$$