Finitely accessible arboreal adjunctions and Hintikka formulae

Luca Reggio & Colin Riba

Department of Computer Science, UCL, UK
LIP, ENS de Lyon, France

Workshop on Resources and Co-Resources

A reformulation of a well-known example

(formulation inspired from Abramsky & Shah (2018, 2021))

Setting

$$\langle \overline{x} \mid \varphi \rangle$$

where

- $\overline{x} = x_1, \dots, x_n$
- φ linearly orders the x_i's (finite conjunction of $(x_i < x_j)$'s)

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Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points. *(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)*

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and symmetrically w.r.t. M and N.

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Corollary

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_{\infty,\omega}(<)$.

Ehrenfeucht-Fraïssé games

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Category of structures: Struct($\boldsymbol{\sigma}$) ($\boldsymbol{\sigma}$ finite relational signature)

- Objects are $\boldsymbol{\sigma}$-structures $M, N, \ldots$
- Morphisms $h\colon M \to N$ preserve relations $R \in \boldsymbol{\sigma}$ (homomorphisms)

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- Positions are spans

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Lemma (Homomorphisms of finitely presented domain)

$$\mathbf{Struct}(\sigma)\left[\langle \overline{x} \mid \varphi \rangle, M\right] \quad \cong \quad [\![\overline{x} \mid \varphi]\!]_M \quad = \quad \{\overline{a} \in M \mid M \models \varphi(\overline{a})\}$$

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or symmetrically w.r.t. M and N.

- Duplicator wins if they can always respond.

Theorem (Karp)

Duplicator has a winning strategy iff M and N are equivalent in $\mathcal{L}_{\infty,\omega}(\boldsymbol{\sigma})$.

(Ehrenfeucht-Fraïssé Theorem: k-rounds games catpure $\mathcal{L}_{\omega,\omega}(\boldsymbol{\sigma})$ with quantifier-depth k)

Game comonads (a particular angle)

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Turn plays into structures

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- Play

$$
\begin{tikzcd}
 & & & & N \\
\langle x_1 \mid \varphi_1 \rangle \arrow[r, tail] \arrow[urrrr, tail, bend left] \arrow[drrrr, tail, bend right] & \langle x_1, x_2 \mid \varphi_1 \wedge \varphi_2 \rangle \arrow[r, tail] \arrow[urrr, tail, bend left] \arrow[drrr, tail, bend right] & \cdots \arrow[r, tail] & \langle x_1, x_2, \ldots, x_n \mid \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \rangle \arrow[ur, tail] \arrow[dr, tail] & \\
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- Plays equipped with pebbles correspond to elements of a σ-structure $R_{\mathbb{P}}(M)$.

$$(\langle \overline{x} \mid \varphi \rangle, \text{Pebbles}) \qquad \text{taken to} \qquad \text{Pebbles}\,(\langle \overline{x} \mid \varphi \rangle) \rightarrowtail M$$

Other examples

- Modal fragment, Hybrid fragment, Guarded fragments, ...
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Adjunctions

- The $R(M)$ are $\boldsymbol{\Sigma}$-structures with a forest order.

$$\begin{tikzcd} \mathcal{A} & \mathbf{Struct}(\boldsymbol{\Sigma}) \arrow[l, bend left, "R"] \end{tikzcd} \qquad (\boldsymbol{\Sigma} \text{ finite signature})$$

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- The $R(M)$ are $\boldsymbol{\Sigma}$-structures with a forest order.
- In each case, R is a right adjoint.
- Comonads on $\mathbf{Struct}(\boldsymbol{\Sigma})$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{Struct}(\boldsymbol{\Sigma}) \arrow[l, bend left, "R"]
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Arboreal categories (a particular angle)

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Motivations. (Abramsky & Reggio (2021, 2023))

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathbf{Struct}(\mathbf{\Sigma}) \arrow[ll, bend left] \end{tikzcd}$$

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Main ideas. (“arboreal quotients” $\mathcal{Q} \subseteq \{\text{epis}\}$, “arboreal embeddings” $\mathcal{M} \subseteq \{\text{monos}\}$)

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$: each morphism f factors as ($e \in \mathcal{Q}$, $m \in \mathcal{M}$)

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- $P \in \mathcal{A}$ is a path when its $\mathcal{M}$-subobjects form a finite chain

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Back-and-forth game $\mathcal{G}(X, Y)$. ($X, Y \in \mathcal{A}$)

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$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{Struct}(\mathbf{\Sigma}) \arrow[l, bend left] \end{tikzcd}$$

- Conditions on $\mathcal{A}$ which yield well-behaved games.

Main ideas. ("arboreal quotients" $\mathcal{Q} \subseteq \{\text{epis}\}$, "arboreal embeddings" $\mathcal{M} \subseteq \{\text{monos}\}$)

- Factorization system $(\mathcal{Q}, \mathcal{M})$ on $\mathcal{A}$: each morphism f factors as ($e \in \mathcal{Q}$, $m \in \mathcal{M}$)

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- Typically, "embeddings" $m \in \mathcal{M}$ are embeddings of $\mathbf{\Sigma}$-structures which are forest morphisms.
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Goal

Sufficient conditions on $R\colon \mathcal{E} \to \mathcal{A}$ so that

$M, N \in \mathcal{E}$ are $\mathcal{L}_{\infty,\omega}$-equivalent $\implies$ $R(M), R(N) \in \mathcal{A}$ are back-and-forth equivalent

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Locally finitely presentable categories. (Gabriel-Ulmer 1971)

- Different characterizations. (see e.g. Adámek-Rosický (1994))

Cartesian theory T:

- Set of implications $\psi \to \varphi$ where ψ, φ built only from atomic formulae, $\top$, $\wedge$ (finite), $\exists!$.

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Definition

$R\colon \mathcal{E} \to \mathcal{A}$ is an arboreal finitely accessible adjunction when the above conditions hold.

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Proof.

- Hintikka formulae for back-and-forth games:
  For each $X \in \mathcal{A}$ and each ordinal α, there is sentence $\Theta_X^\alpha \in \mathcal{L}_{\infty,\omega}(\mathbf{\Gamma})$ such that

$$Y \models \Theta_X^\alpha \qquad \Longleftrightarrow \qquad \text{the initial position of } \mathcal{G}(X, Y) \text{ has rank } \alpha$$

- Functorial semantics and Yoneda Lemma. (Syntactic categories for cartesian theories)

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Proof.

- The finitary right adjoint $R\colon \mathbf{Mod}(\mathbf{T}) \to \mathbf{Mod}(\mathbf{U})$ induces an interpretation

$$\mathcal{L}_{\kappa,\omega}(\mathbf{\Gamma}) \quad \longrightarrow \quad \mathcal{L}_{\kappa,\omega}(\mathbf{\Sigma}) \qquad (\kappa \text{ regular cardinal})$$

Main result

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Let

- $\mathcal{A} \cong$ **Mod**(**U**) with **U** cartesian theory of signature $\mathbf{\Gamma}$.
- $\mathcal{E} \cong$ **Mod**(**T**) with **T** cartesian theory of signature $\mathbf{\Sigma}$. (e.g. $\mathcal{E} =$ **Struct**($\mathbf{\Sigma}$))

Main result

Consider an arboreal finitely accessible adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend left, "R"] \end{tikzcd}$$

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Definition (Detection of path embeddings)

$L \dashv R \colon \mathcal{E} \to \mathcal{A}$ detects path embeddings when

$f\colon P \to X$ "arboreal embedding" in $\mathcal{A}$ $\iff$ $L(f)$ embedding of structures in $\mathcal{E}$

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Theorem (Reggio & R)

Assume $L \dashv R \colon \mathcal{E} \to \mathcal{A}$ detects path embeddings.
If $M, N \in \mathcal{E}$ are equivalent in $\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma})$, then $R(M), R(N)$ are back-and-forth equivalent in $\mathcal{A}$.

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(1) In $\mathbf{Mod}(\mathbf{T})$, embeddings of finitely presentable domain are $\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma})$-definable.

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$$f \colon P \to X \text{ “arboreal embedding” in } \mathcal{A} \quad \Longleftrightarrow \quad L(f) \text{ embedding of structures in } \mathcal{E}$$

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Proof.

(1) In $\mathbf{Mod}(\mathbf{T})$, embeddings of finitely presentable domain are $\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma})$-definable.
(2) The (finitary) left adjoint $L \colon \mathbf{Mod}(\mathbf{U}) \to \mathbf{Mod}(\mathbf{T})$ induces a formula translation

$$\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma}) \longrightarrow \mathcal{L}_{\infty,\omega}(\mathbf{\Gamma})$$ (Hodges' word-constructions (1974, 1975))

Discussion

Consider an arboreal finitely accessible adjunction which detects path embeddings

($\boldsymbol{\sigma}$ finite relational signature)

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Example

- Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points. (e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

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- Many non-isomorphic $\mathcal{L}_{\infty,\omega}$-equivalent structures.

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Game comonad for MSO (Jackl, Marsden & Shah, 2022)

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not $\mathbf{MSO}(<)$-equivalent.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- General conditions on $R\colon \mathcal{E} \to \mathcal{A}$ for

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- Restricts to finite games and finitary logic. (Under suitable conditions)
- Covers different examples

(Ehrenfeucht-Fraïssé and pebble games, modal and hybrid logics)
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Thanks for your attention!