

Finely accessible arboreal adjunctions and Hintikka formulae

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Workshop on Resources and Co-Resources

A reformulation of a well-known example

(formulation inspired from Abramsky & Shah (2018, 2021))

Setting

$$\langle \bar{x} \mid \varphi \rangle$$

where

▶ $\bar{x} = x_1, \dots, x_n$

▶ φ linearly orders the x_i 's

(finite conjunction of $(x_i < x_j)$'s)

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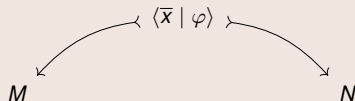
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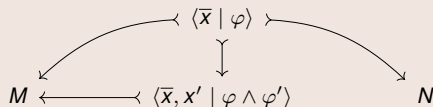
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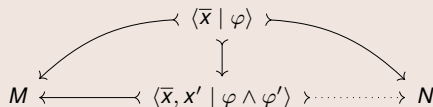
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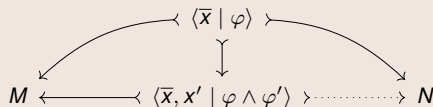
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Corollary

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_{\infty, \omega}(<)$.

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Category of structures: $\mathbf{Struct}(\sigma)$

- ▶ Objects are σ -structures M, N, \dots
- ▶ Morphisms $h: M \rightarrow N$ preserve relations $R \in \sigma$

(σ finite relational signature)

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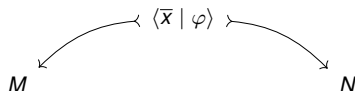
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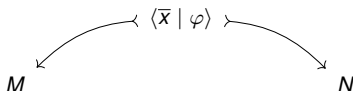
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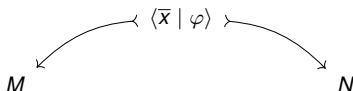
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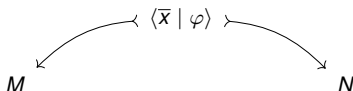
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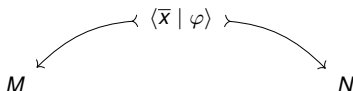
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(played by Spoiler and Duplicator)



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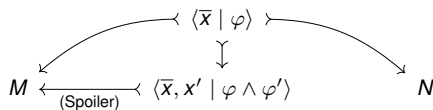
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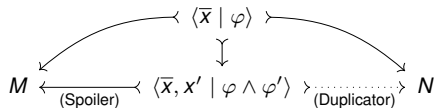
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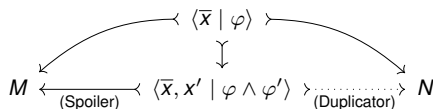
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- ▶ Duplicator wins if they can always respond.

Theorem (Karp)

Duplicator has a winning strategy iff M and N are equivalent in $\mathcal{L}_{\infty, \omega}(\sigma)$.

(Ehrenfeucht-Fraïssé Theorem: k -rounds games capture $\mathcal{L}_{\omega, \omega}(\sigma)$ with quantifier-depth k)

Game comonads (a particular angle)

Main idea:

Turn plays into structures

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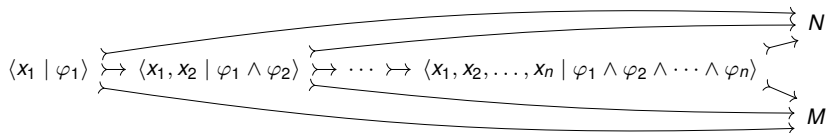
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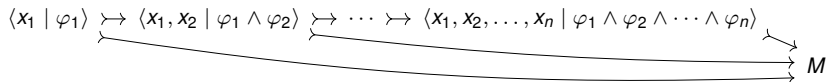
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(M^+ = n.e. finite words on M)

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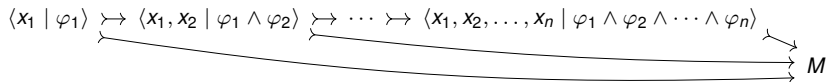
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Pebble games

(Abramsky, Dawar & Wang (2017), Abramsky & Shah (2018, 2021))

- ▶ Plays equipped with **pebbles** correspond to elements of a σ -structure $R_{\text{P}}(M)$.

$(\langle \bar{x} \mid \varphi \rangle, \text{Pebbles})$ taken to Pebbles $(\langle \bar{x} \mid \varphi \rangle) \rightsquigarrow M$

Other examples

- ▶ Modal fragment, Hybrid fragment, Guarded fragments, ...
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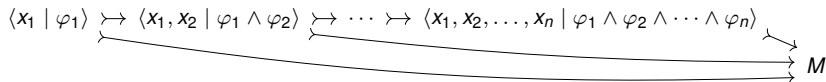
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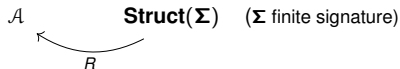
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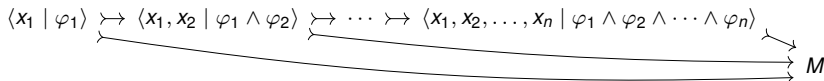
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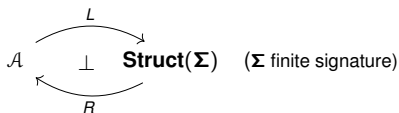
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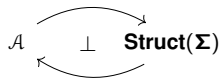
- ▶ The $R(M)$ are Σ -structures with a forest order.
- ▶ In each case, R is a right adjoint.
- ▶ Comonads on **Struct**(Σ).



Arboreal categories (a particular angle)

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Motivations.



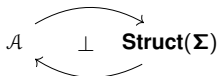
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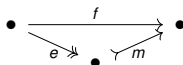


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(“arboreal quotients” $\mathcal{Q} \subseteq \{\text{epis}\}$, “arboreal embeddings” $\mathcal{M} \subseteq \{\text{monos}\}$)

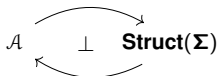
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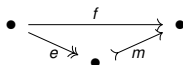


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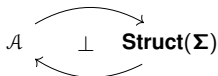


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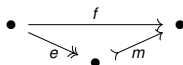


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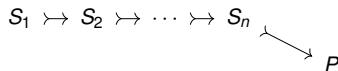
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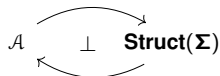
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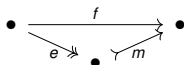


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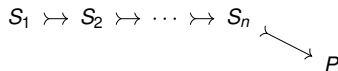
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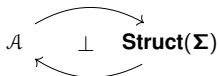
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Arboreal categories (a particular angle)

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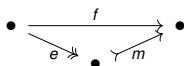


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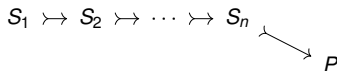
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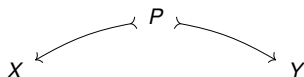


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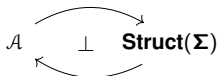
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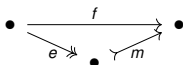


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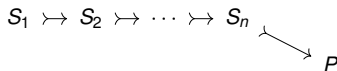
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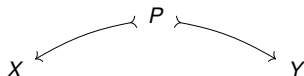
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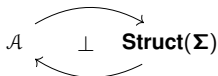
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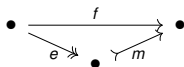


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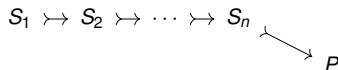
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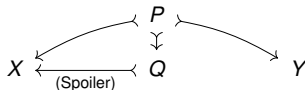


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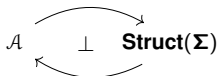
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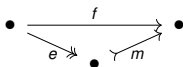


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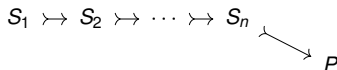
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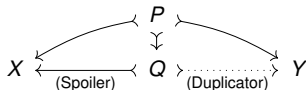


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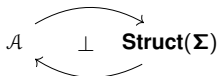


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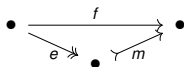


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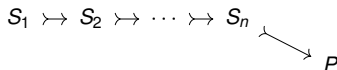
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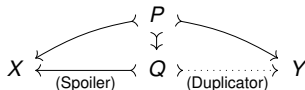
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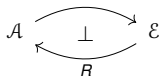
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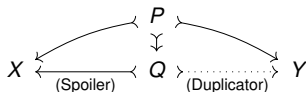
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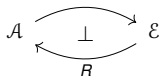
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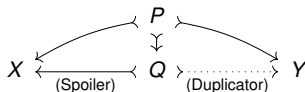
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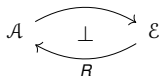
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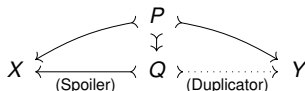
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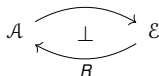
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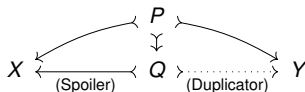
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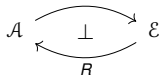
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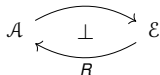
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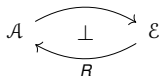
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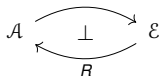
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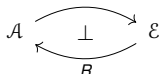
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Locally finitely presentable categories.

- ▶ Different characterizations.

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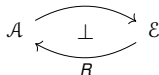
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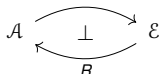
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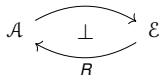
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Proof.

- ▶ Hintikka formulae for back-and-forth games:

For each $X \in \mathcal{A}$ and each ordinal α , there is sentence $\Theta_X^\alpha \in \mathcal{L}_{\infty, \omega}(\Gamma)$ such that

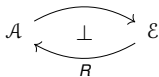
$$Y \models \Theta_X^\alpha \iff \text{the initial position of } \mathfrak{G}(X, Y) \text{ has rank } \alpha$$

- ▶ Functorial semantics and Yoneda Lemma.

(Syntactic categories for cartesian theories)

Results: Arboreal finitely accessible adjunctions

Consider an arboreal finitely accessible adjunction



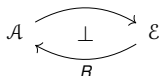
Let

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Assumption (Definable path embeddings)

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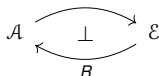
$$X \models \text{Emb}_P(\bar{a}) \iff \bar{a} \in X \text{ induces an "arboreal embedding" } P \hookrightarrow X$$

Corollary (Reggio & R)

If $M, N \in \mathcal{E}$ are equivalent in $\mathcal{L}_{\infty,\omega}(\Sigma)$, then $R(M), R(N)$ are back-and-forth equivalent in \mathcal{A} .

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Proof.

- ▶ The finitary right adjoint $R: \mathbf{Mod}(\mathbf{T}) \rightarrow \mathbf{Mod}(\mathbf{U})$ induces an interpretation

$$\mathcal{L}_{\kappa, \omega}(\Gamma) \longrightarrow \mathcal{L}_{\kappa, \omega}(\Sigma) \quad (\kappa \text{ regular cardinal})$$

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$$\mathcal{A} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathcal{E}$$

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$L \dashv R: \mathcal{E} \rightarrow \mathcal{A}$ detects path embeddings when

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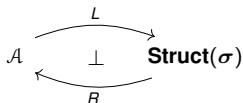
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Discussion

Consider an arboreal finitely accessible adjunction which detects path embeddings

(σ finite relational signature)



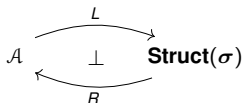
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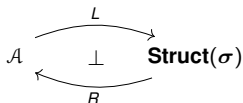
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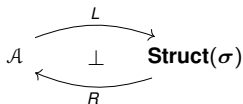
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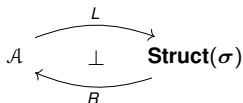
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Game comonad for MSO

(Jackl, Marsden & Shah, 2022)

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Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

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