Finitely accessible arboreal adjunctions and Hintikka formulae

Luca Reggio & Colin Riba

Department of Computer Science, UCL, UK
LIP, ENS de Lyon, France

Structure meets Power 2023

A reformulation of a well-known example

(formulation inspired from Abramsky & Shah (2018, 2021))

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Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points. *(e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)*

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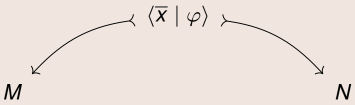

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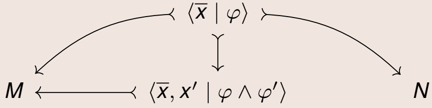

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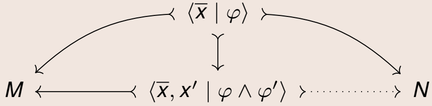

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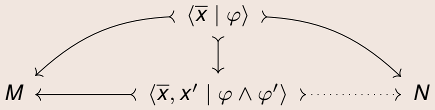

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Corollary

$(M, <_M)$ and $(N, <_N)$ are equivalent in $\mathcal{L}_{\infty,\omega}(<)$.

Ehrenfeucht-Fraïssé games

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Category of structures: Struct(σ) ($\boldsymbol{\sigma}$ finite relational signature)

- Objects are $\boldsymbol{\sigma}$-structures $M, N, \ldots$
- Morphisms $h \colon M \to N$ preserve relations $R \in \boldsymbol{\sigma}$ (homomorphisms)

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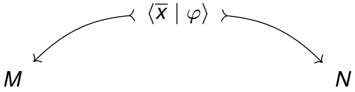

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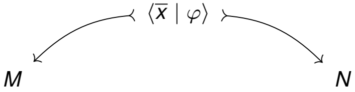

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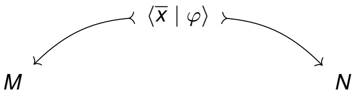

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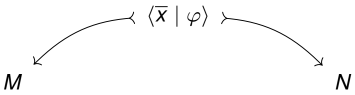

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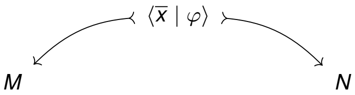

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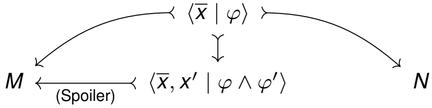

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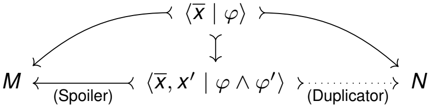

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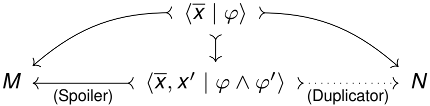

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- Duplicator wins if they can always respond.

Theorem (Karp)

Duplicator has a winning strategy iff M and N are equivalent in $\mathcal{L}_{\infty,\omega}(\boldsymbol{\sigma})$.

(Ehrenfeucht-Fraïssé Theorem: k-rounds games catpure $\mathcal{L}_{\omega,\omega}(\boldsymbol{\sigma})$ with quantifier-depth k)

Game comonads (a particular angle)

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Turn plays into structures

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Ehrenfeucht-Fraïssé games (Abramsky & Shah (2018, 2021))

- Play

$$\langle x_1 \mid \varphi_1 \rangle \rightarrowtail \langle x_1, x_2 \mid \varphi_1 \wedge \varphi_2 \rangle \rightarrowtail \cdots \rightarrowtail \langle x_1, x_2, \ldots, x_n \mid \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \rangle$$

Each of these structures has an embedding $\rightarrowtail N$ and an embedding $\rightarrowtail M$.

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- Plays equipped with pebbles correspond to elements of a σ-structure $R_{\mathbb{P}}(M)$.

$$(\langle \overline{x} \mid \varphi \rangle, \text{Pebbles}) \qquad \text{taken to} \qquad \text{Pebbles}\,(\langle \overline{x} \mid \varphi \rangle) \rightarrowtail M$$

Other examples

- Modal fragment, Hybrid fragment, Guarded fragments, . . .
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- The $R(M)$ are $\mathbf{\Sigma}$-structures with a forest order.
- In each case, R is a right adjoint.
- Comonads on $\mathbf{Struct}(\mathbf{\Sigma})$.

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Struct}(\mathbf{\Sigma}) \quad (\bot) \qquad (\mathbf{\Sigma} \text{ finite signature})$$

Arboreal categories (a particular angle)

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Motivations. (Abramsky & Reggio (2021, 2023))

$$\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{Struct}(\boldsymbol{\Sigma})$$

- Conditions on $\mathcal{A}$ which yield well-behaved games.

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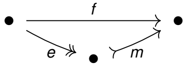

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Back-and-forth game $\mathcal{G}(X, Y)$. ($X, Y \in \mathcal{A}$)

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Arboreal categories (a particular angle)

Motivations. (Abramsky & Reggio (2021, 2023))

$$\mathcal{A} \underset{}{\overset{}{\rightleftarrows}} \mathbf{Struct}(\mathbf{\Sigma}) \quad (\bot)$$

- Conditions on $\mathcal{A}$ which yield well-behaved games.

Main ideas. (“arboreal quotients” $\mathcal{Q} \subseteq \{\text{epis}\}$, “arboreal embeddings” $\mathcal{M} \subseteq \{\text{monos}\}$)

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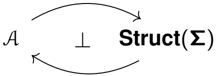

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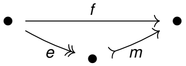

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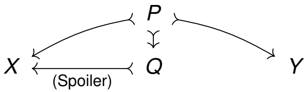

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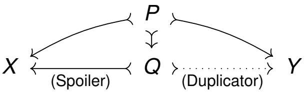

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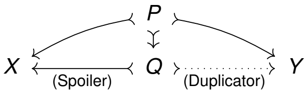

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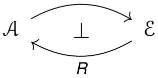

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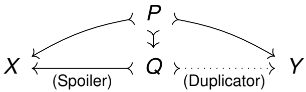

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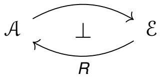

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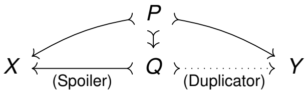

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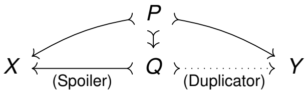

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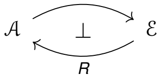

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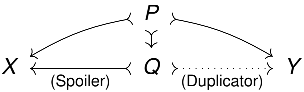

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Sufficient conditions on $R\colon\ \mathcal{E} \to \mathcal{A}$ so that

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- Different characterizations. (see e.g. Adámek-Rosický (1994))

Cartesian theory T:

- Set of implications $\psi \to \varphi$ where ψ, φ built only from atomic formulae, $\top$, $\wedge$ (finite), $\exists!$.

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Definition

$R\colon \mathcal{E} \to \mathcal{A}$ is an arboreal finitely accessible adjunction when the above conditions hold.

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Proof.

- Hintikka formulae for back-and-forth games:
  For each $X \in \mathcal{A}$ and each ordinal α, there is sentence $\Theta^{\alpha}_X \in \mathcal{L}_{\infty,\omega}(\mathbf{\Gamma})$ such that

$$Y \models \Theta^{\alpha}_X \qquad \Longleftrightarrow \qquad \text{the initial position of } \mathcal{G}(X, Y) \text{ has rank } \alpha$$

- Functorial semantics and Yoneda Lemma. (Syntactic categories for cartesian theories)

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Proof.

- The finitary right adjoint $R\colon \mathbf{Mod}(\mathbf{T}) \to \mathbf{Mod}(\mathbf{U})$ induces an interpretation

$$\mathcal{L}_{\kappa,\omega}(\mathbf{\Gamma}) \longrightarrow \mathcal{L}_{\kappa,\omega}(\mathbf{\Sigma})$$

(κ regular cardinal)

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Consider an arboreal finitely accessible adjunction

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{E} \quad (L \dashv R)$$

Let

- $\mathcal{A} \cong \mathbf{Mod}(\mathbf{U})$ with $\mathbf{U}$ cartesian theory of signature $\mathbf{\Gamma}$.
- $\mathcal{E} \cong \mathbf{Mod}(\mathbf{T})$ with $\mathbf{T}$ cartesian theory of signature $\mathbf{\Sigma}$. (e.g. $\mathcal{E} = \mathbf{Struct}(\mathbf{\Sigma})$)

Definition (Detection of path embeddings)

$L \dashv R \colon \mathcal{E} \to \mathcal{A}$ detects path embeddings when

$f \colon P \to X$ "arboreal embedding" in $\mathcal{A}$ $\iff$ $L(f)$ embedding of structures in $\mathcal{E}$

Theorem (Reggio & R)

Assume $L \dashv R \colon \mathcal{E} \to \mathcal{A}$ detects path embeddings.
If $M, N \in \mathcal{E}$ are equivalent in $\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma})$, then $R(M), R(N)$ are back-and-forth equivalent in $\mathcal{A}$.

Proof.

(1) In $\mathbf{Mod}(\mathbf{T})$, embeddings of finitely presentable domain are $\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma})$-definable.

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(2) The (finitary) left adjoint $L \colon \mathbf{Mod}(\mathbf{U}) \to \mathbf{Mod}(\mathbf{T})$ induces a formula translation

$$\mathcal{L}_{\infty,\omega}(\mathbf{\Sigma}) \longrightarrow \mathcal{L}_{\infty,\omega}(\mathbf{\Gamma})$$

(Hodges' word-constructions (1974, 1975))

Discussion

Consider an arboreal finitely accessible adjunction which detects path embeddings

($\boldsymbol{\sigma}$ finite relational signature)

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Example

- Let $(M, <_M)$ and $(N, <_N)$ be dense linear orders without end points. (e.g. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$)

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Game comonad for MSO (Jackl, Marsden & Shah, 2022)

- $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are not **MSO**$(<)$-equivalent.

Conclusion and future work

Toward a structure theory of game comonads via arboreal categories.

- General conditions on $R\colon \mathcal{E} \to \mathcal{A}$ for

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- Restricts to finite games and finitary logic. (Under suitable conditions)
- Covers different examples

(Ehrenfeucht-Fraïssé and pebble games, modal and hybrid logics)
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Thanks for your attention!