Capturing Quantum Isomorphism in the Kleisli Category of the Quantum Monad

Amin Karamlou

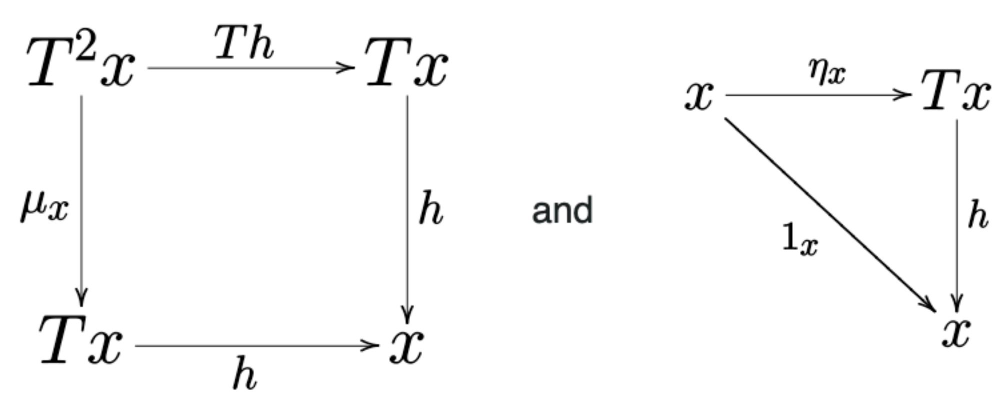
Monads

A **monad** is a triple (T, η, μ) where:

1. $T\colon C \to C$ is an endofunctor.
2. The **unit** $\eta\colon id_T \to T$ is a natural transformation.
3. The **multiplication** $\mu\colon T^2 \to T$ is a natural transformation.

And the following equations hold:

$$\mu \circ T\eta = \mu \circ \eta T = id_T; \quad \mu \circ T\mu = \mu \circ \mu T$$



Monads

Monads model effectful computation e.g. nondeterminism, probabilities.

An effectful computation from A to B is represented as a morphism $A \to MB$ and can be composed in the **Kleisli** category $Kl(M)$ of a monad M:

- $Obj(Kl(M)) = Obj(C)$
- $Hom_{Kl(M)}(A, B) = Hom(A, MB)$
- $id_x = \eta_x$
- $g \circ f = X \xrightarrow{f} MY \xrightarrow{g*} MZ$ where:
    - $f : X \to MY$ and $g : Y \to MZ$
    - $g* = \mu_z \circ Mg$

Example: Distribution Monad

The **distribution monad** $(\mathscr{D}, \eta, \mu)$ on the category of sets and functions is given by:

1. $\mathscr{D}X$ is the set of all functions of the form $\psi : X \to [0,1]$ where $\Sigma_{x \in X} \psi(x) = 1$.

2. $\eta_X(x) = 1.x$

3. $\mu_X(\psi)(x) = \Sigma_{\phi \in \mathscr{D}X} \psi(\phi) \,.\, \phi(x)$

Example: Distribution Monad

- Morphisms $X \xrightarrow{\mathbf{A}} \mathscr{D}Y$ in the kleisli category of $\mathscr{D}$ are **row stochastic maps.**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

All rows add to 1

Example: Quantum Tensor Monad

- An (orthogonal) **projection matrix** is a square matrix $\mathbf{P}$ satisfying $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^*$
- We write $proj(d)$ for the set of $d \times d$ projection matrices.

The (graded) **quantum tensor monad** $(\mathcal{Q}_d, \eta, \mu^{d,d'})$ on the category of sets and functions is given by:

1. $\mathcal{Q}_d X$ is the set of all functions of the form $\psi : X \to proj(d)$ where $\Sigma_{x \in X} \psi(g) = I_d$.
2. $\eta_X(x) = \delta_x \in \mathcal{Q}_1 X$ where for $x \neq x'$: $\begin{cases} \delta_a(a) = I_1 \\ \delta_a(a') = \mathbf{0}_1 \end{cases}$
3. $\mu_X^{d,d'}(\psi)(x) = \Sigma_{\phi \in \mathcal{Q}_{d'} X} \psi(\phi) \otimes \phi(x)$

Example: Quantum Tensor Monad

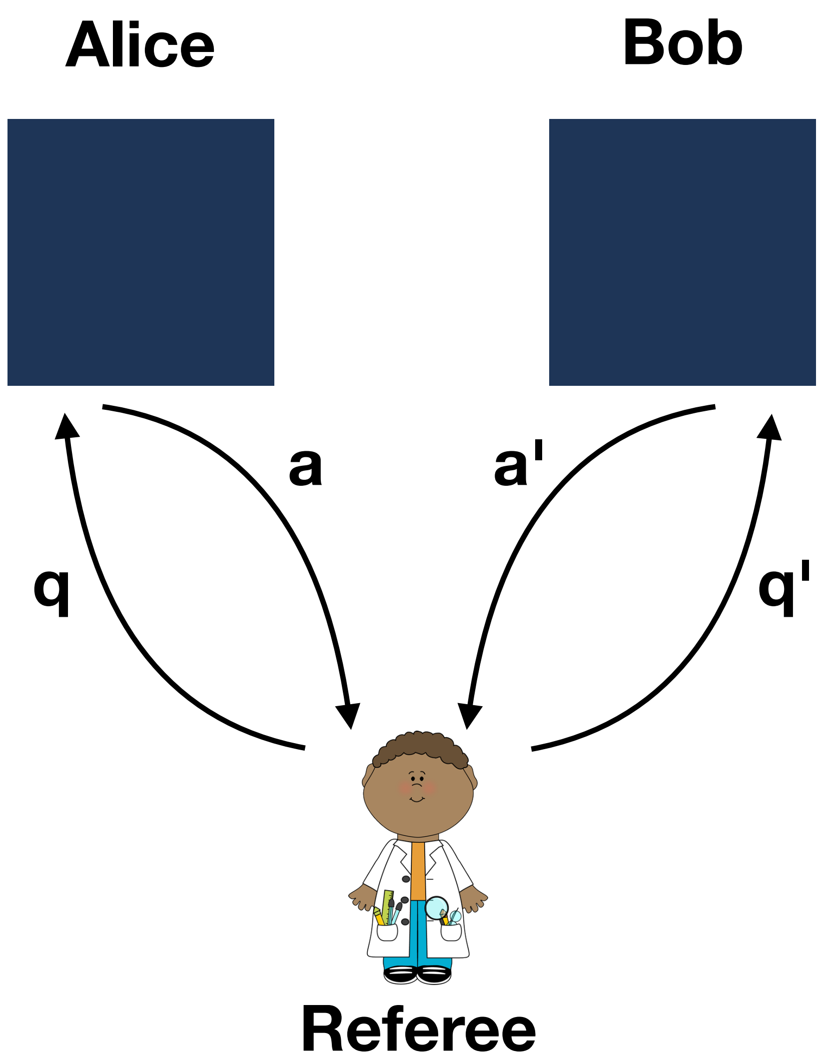
- We shall call Morphisms $X \xrightarrow{\mathbf{A}} \mathcal{Q}_d Y$ in the kleisli category of $\mathcal{Q}_d$ **row projective permutation matrices.**

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1s} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qs} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

All rows add to I_d

- Think of this as a variant of the distribution monad where probabilities are replaced with projectors. Each row thus represents a PVM.

Non-local game

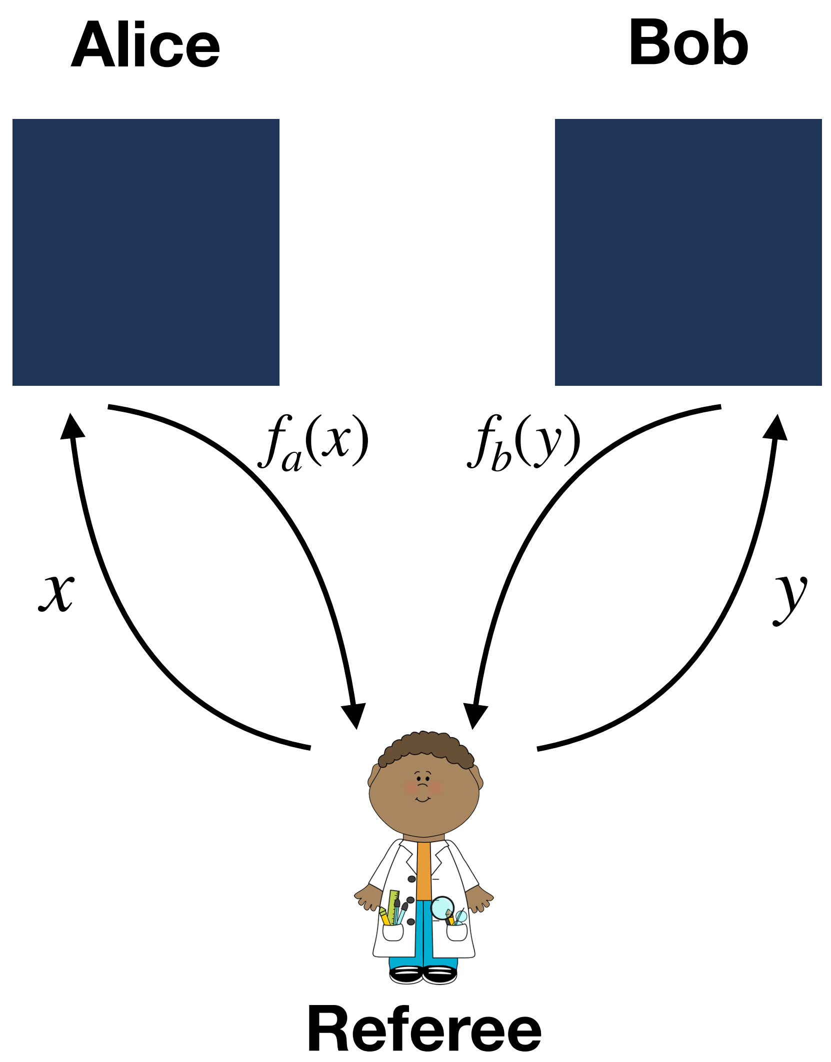


1. Referee sends a question to each player
2. Players answer **without communicating**
3. Win if answers satisfy some predefined conditions.

Note that players Can agree on a strategy beforehand.

We focus only on perfect strategies.

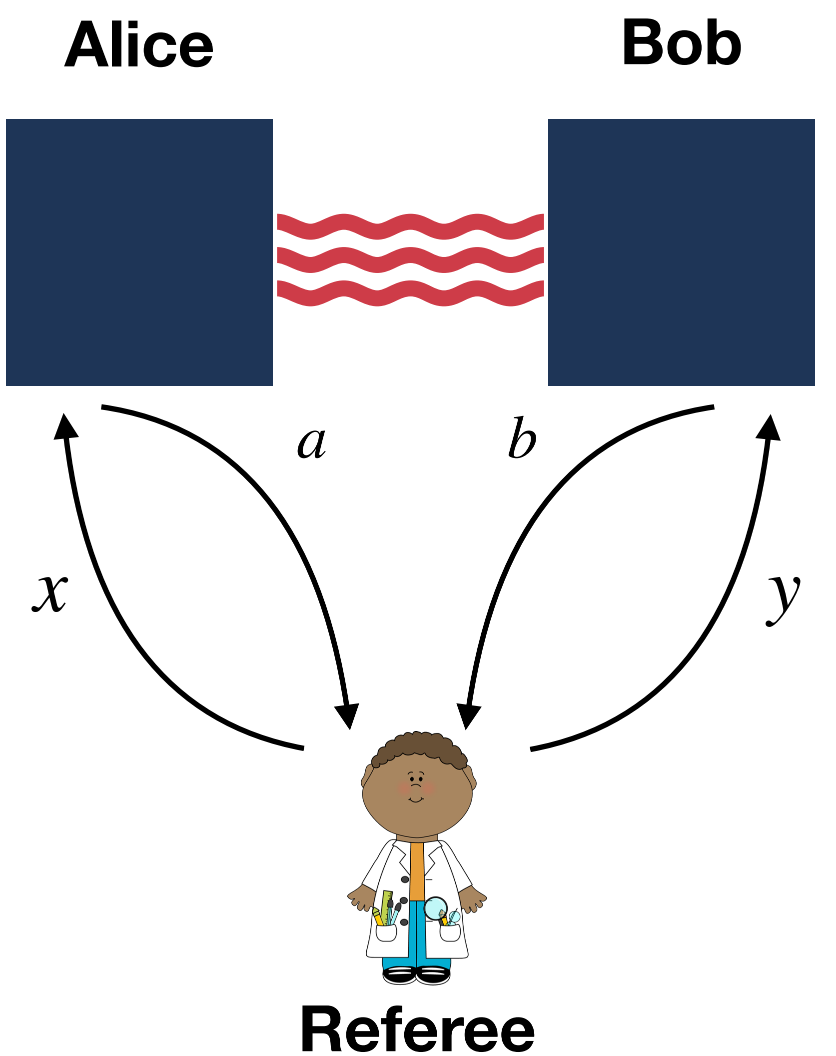
Classical Strategies



- Deterministic functions f_a and f_b.

$$p(f_a(x), f_b(y) \,|\, x, y) = 1$$

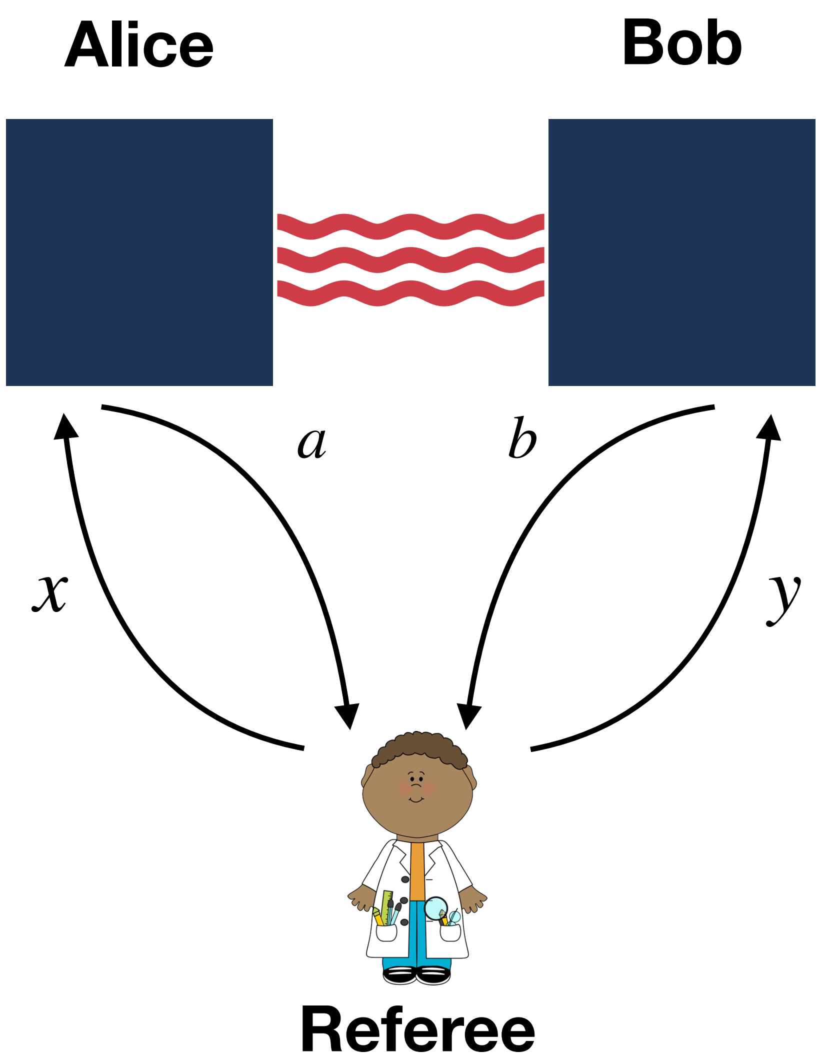
Quantum Tensor Strategies



- Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$
- Shared entangled state $\psi \in \mathscr{H}_A \otimes \mathscr{H}_B$
- For any inputs x, y, POVMs $\{A_{x,a}\}_a$, $\{B_{x,b}\}_b$ acting on $\mathscr{H}_A$ and $\mathscr{H}_B$

$$p(a, b \,|\, x, y) = \psi^\dagger A_{x,a} \otimes B_{y,b} \psi$$

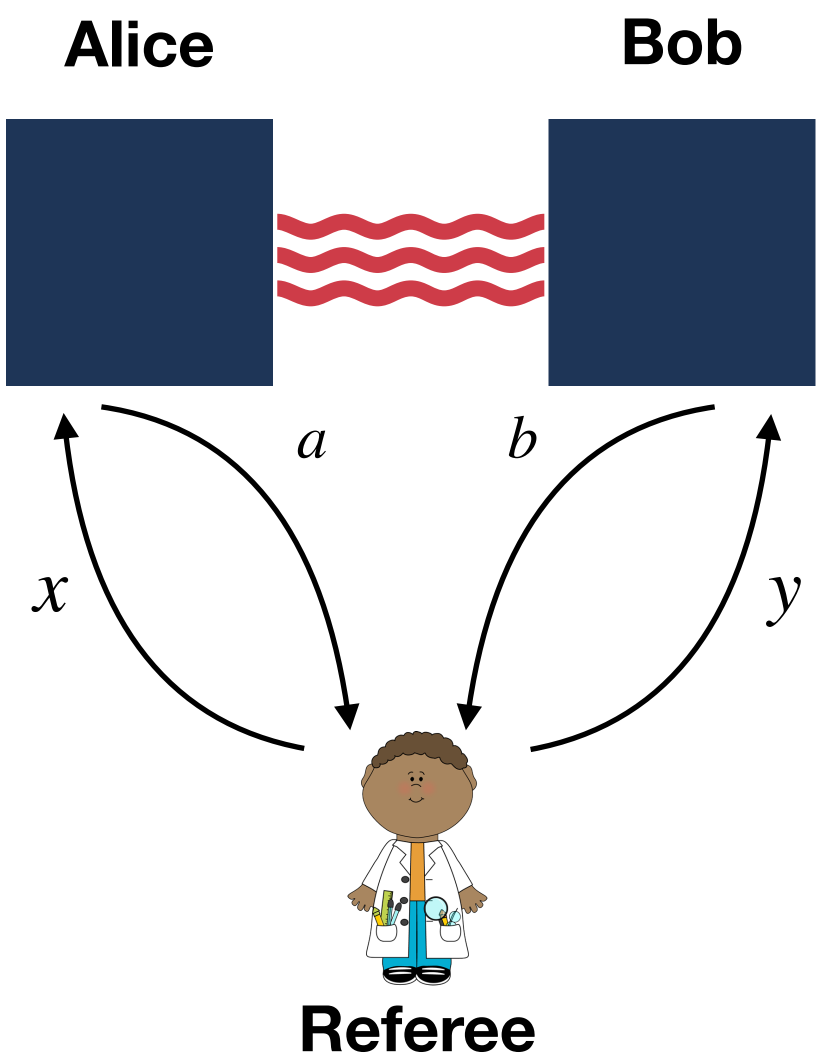
Quantum Commuting Strategies



- Hilbert space $\mathscr{H}$
- Shared entangled state $\psi \in \mathscr{H}$
- For any inputs x, y, POVMs $\{A_{x,a}\}_a$, $\{B_{x,b}\}_b$, acting on $\mathscr{H}$
- $A_{x,a}$ and $B_{y,b}$ commute for all x, a, y, b.

$$p(a, b \,|\, x, y) = \psi^\dagger A_{x,a} B_{y,b} \psi$$

Non-Signalling Strategies

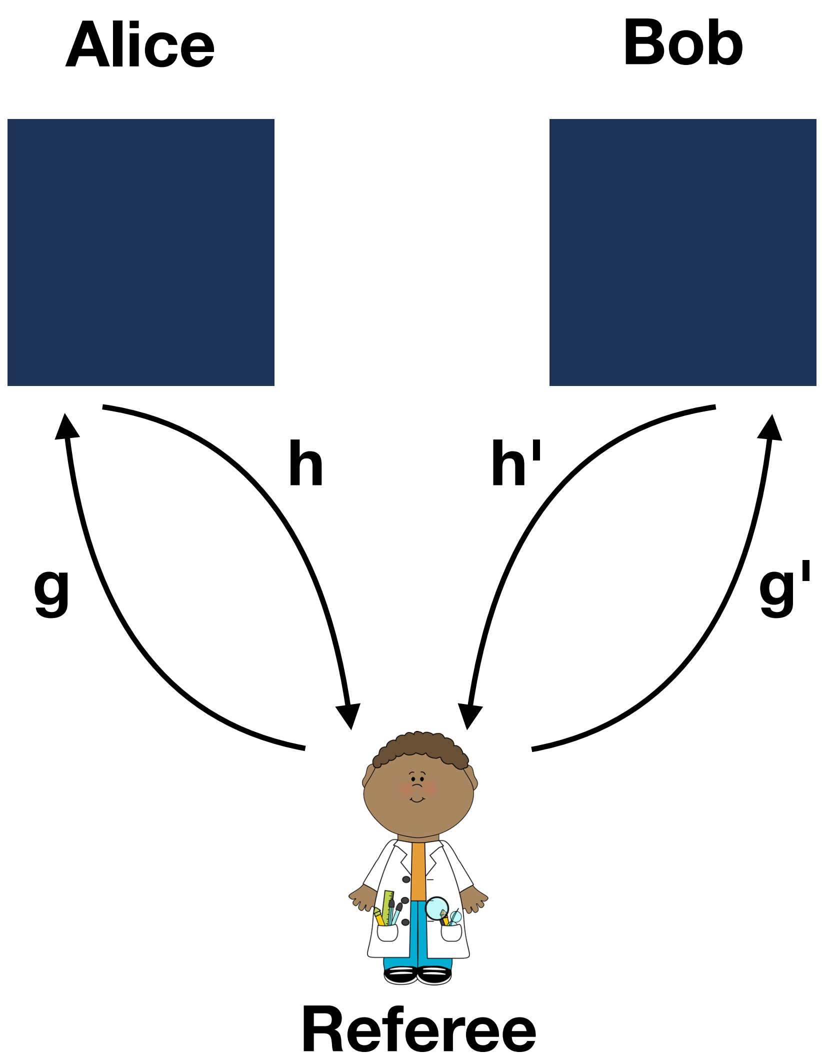


- Any strategy where:

$$\sum_{y_b} p(y_a, y_b \,|\, x_a, x_b) = \sum_{y_b} p(y_a, y_b \,|\, x_a, x_b') \;\forall x_a, y_a, x_b, x_b'$$

- Most general class of strategies with no communication.

(G, H)-Homomorphism Game

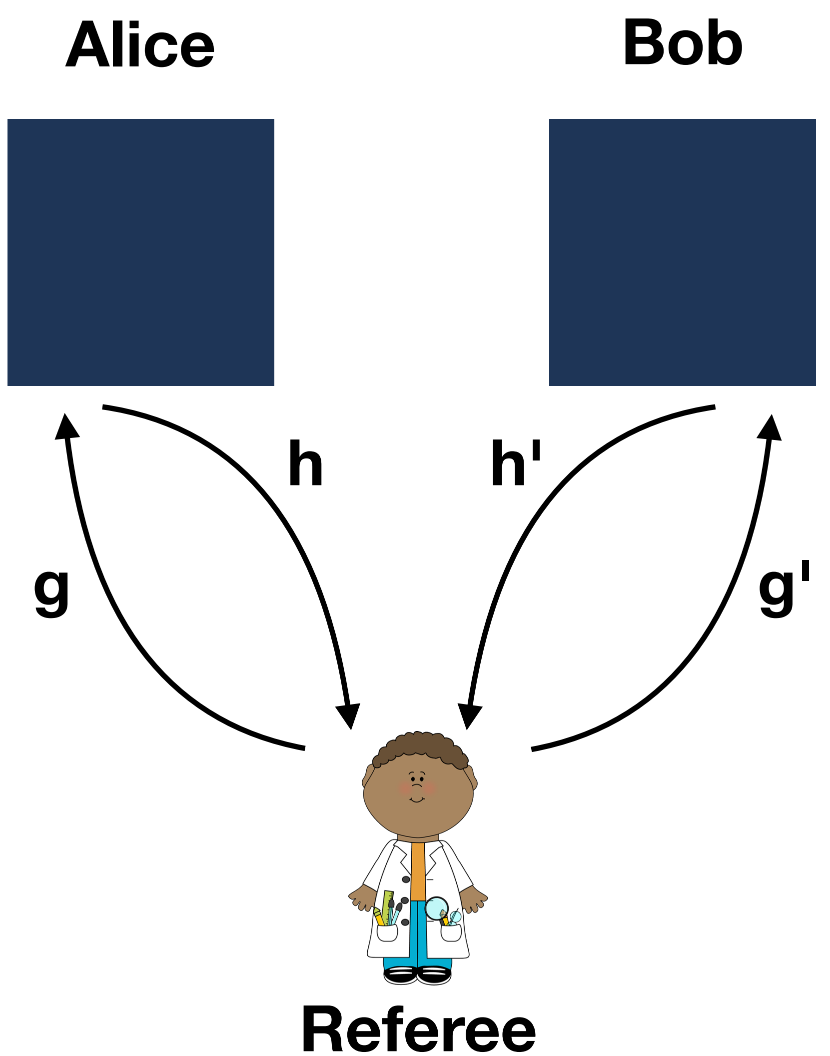


Intuition: Alice and Bob want to convince referee that $G \to H$

1. Referee sends them both vertices of G
2. They respond with vertices of H
3. Win if adjacency and equality preserved

We write $G \xrightarrow{t} H$ whenever the game has a winning t-strategy for $t \in \{c, *, co, ns\}$

(G, H)-Isomorphism Game

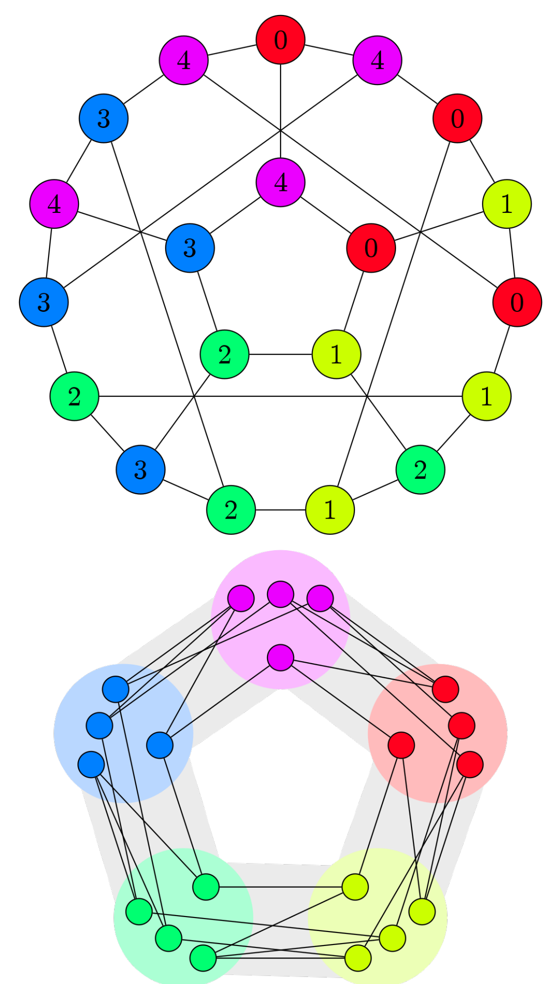


Intuition: Alice and Bob want to convince referee that $G \cong H$

1. Referee sends vertices from either graph
2. Players respond with vertices from other graph
3. Win if vertex relationships preserved

We write $G \overset{t}{\cong} H$ whenever the game has a winning t-strategy for $t \in \{c, *, co, ns\}$

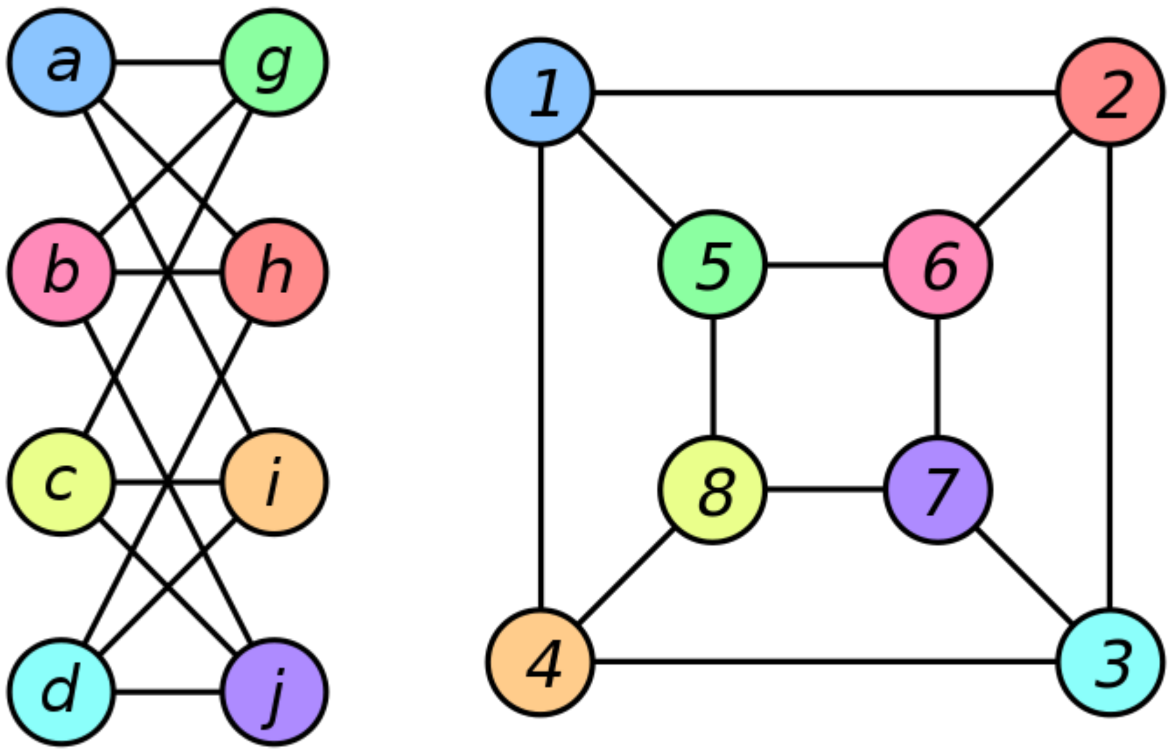
Classical Strategies



[1]: The (G,H)-Homomorphism game admits a perfect classical strategy iff $G \to H$.

[1] Mančinska, Laura, and David E. Roberson. "Quantum homomorphisms." *Journal of Combinatorial Theory, Series B* 118 (2016): 228-267.

Classical Strategies



[1]: The (G,H)-isomorphism game admits a perfect classical strategy iff $G \cong H$.

[1] Atserias, Albert, et al. "Quantum and non-signalling graph isomorphisms." *Journal of Combinatorial Theory, Series B* 136 (2019): 289-328.

Quantum Tensor Strategies

[1]: $G \xrightarrow{*} H$ iff there exists projectors $\mathbf{A}_{gh}$ satisfying

1. $\sum_{h \in V(H)} \mathbf{A}_{gh} = I \; \forall g \in V(G)$
2. $(g \sim g' \,\&\, h \nsim h') \implies \mathbf{A}_{gh}\mathbf{A}_{g'h'} = 0$

- Condition 1 is equivalent to the existence of a row projective permutation map.

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1s} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \cdots & \mathbf{A}_{qs} \end{bmatrix}$$

- Condition 2 places constraints on elements of the block matrix

[1] Mančinska, Laura, and David E. Roberson. "Quantum homomorphisms." *Journal of Combinatorial Theory, Series B* 118 (2016): 228-267.

Quantum Tensor Strategies

[1]: $G \overset{*}{\cong} H$ iff there exists projectors $\mathbf{A}_{gh}$ satisfying

1. $\sum_{h \in V(H)} \mathbf{A}_{gh} = I \, \forall g \in V(G)$

2. $\sum_{g \in V(G)} \mathbf{A}_{gh} = I \, \forall h \in V(H)$

3. $(g \sim g' \,\&\, h \nsim h') \vee (g \nsim g' \,\&\, h \sim h') \implies \mathbf{A}_{gh}\mathbf{A}_{g'h'} = 0 \implies \mathbf{A}_{gh}\mathbf{A}_{g'h'} = 0$

- Condition 2 enforces that the columns of the matrix also add up to the identity. This structure is sometimes referred to as a **projective permutation matrix.**

[1] Mančinska, Laura, and David E. Roberson. "Quantum homomorphisms." *Journal of Combinatorial Theory, Series B* 118 (2016): 228-267.

Quantum Monad on Graphs

- The Quantum monad $\mathbb{Q}_d$ on the category of graphs and graph homomorphisms is defined as follows:

1. It acts exactly the same way as $\mathcal{Q}_d$ on the set of vertices of the graph.

2. $\psi \sim \psi'$ if whenever $g \nsim g'$ in G then $\psi(g) \,.\, \psi(g') = 0$.

[1]: $G \xrightarrow{*} H$ iff there exists a kleisli morphism $G \to \mathbb{Q}_d H$

- So do Kleisli isomorphisms correspond to quantum tensor isomorphism? Nope!

$G \cong H$ iff there exists a kleisli isomorphism $G \cong_{kl(\mathbb{Q}_d)} H$

[1] Abramsky, Samson, et al. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Quantum Monad on Graphs

- To capture quantum isomorphism we need an intermediate notion of equivalence.

- We write $G \rightleftarrows_{kl(\mathbb{Q}_d)} H$ iff there exists morphisms $G \xrightarrow{\mathbf{A}} \mathbb{Q}_d H$ and $H \xrightarrow{\mathbf{B}} \mathbb{Q}_d G$ such that $\mathbf{A} = \mathbf{B}^\dagger$

[Theorem, MR16] $G \xrightarrow{*} H$ iff $G \rightleftarrows_{kl(\mathbb{Q}_d)} H$

- This is somewhat reminiscent of a result in [1]:

We now define the relation $A \rightleftarrows_k^- B$ if there are co-Kleisli arrows $f : \mathbb{T}_k A \longrightarrow B$ and $g : \mathbb{T}_k B \longrightarrow A$ such that $S_f^- = S_g$.

Theorem 20. *For all finite structures A and B:*

$$A \rightleftarrows_k^- B \iff A \equiv^k B.$$

---

[1] Abramsky, Samson, Anuj Dawar, and Pengming Wang. "The pebbling comonad in finite model theory." *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2017.

Quantum Commuting Strategies

[1]: $G \overset{co}{\to} H$ iff there exists a unital C*-algebra $\mathscr{A}$ and projections $u_{gh} \in \mathscr{A}$ satisfying

1. $\sum_{h \in V(H)} u_{gh} = I \; \forall g \in V(G)$
2. $(g \sim g' \;\&\; h \nsim h') \implies u_{gh}u_{g'h'} = 0$

- Again we can arrange these projectors into a matrix. We will call a matrix satisfying condition 1 above a **row quantum permutation matrix.**

$$u = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1M} \\ u_{21} & u_{22} & \cdots & u_{2M} \\ & \vdots & & \\ u_{N1} & u_{2N} & \cdots & u_{NM} \end{pmatrix}$$

[1] Ortiz, Carlos M., and Vern I. Paulsen. "Quantum graph homomorphisms via operator systems." *Linear Algebra and its Applications* 497 (2016): 23-43.

Quantum Commuting Strategies

[1]: $G \overset{co}{\cong} H$ iff there exists a unital C*-algebra $\mathscr{A}$ and projections $u_{gh} \in \mathscr{A}$ satisfying

1. $$\sum_{h \in V(H)} u_{gh} = I \; \forall g \in V(G)$$
2. $$\sum_{g \in V(G)} u_{gh} = I \; \forall h \in V(H)$$
3. $(g \sim g' \;\&\; h \nsim h') \vee (g \nsim g' \;\&\; h \sim h') \implies u_{gh}u_{g'h'} = 0$

- Here we have what is known as a **quantum permutation matrix** or **magic unitary.**

$$u = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1M} \\ u_{21} & u_{22} & \cdots & u_{2M} \\ & \vdots & & \\ u_{N1} & u_{2N} & \cdots & u_{NM} \end{pmatrix}$$

[1] Atserias, Albert, et al. "Quantum and non-signalling graph isomorphisms." *Journal of Combinatorial Theory, Series B* 136 (2019): 289-328.

Quantum Commuting Endofunctor

- Fix an arbitrary unital C*-algebra $\mathscr{A}$ and write $proj(\mathscr{A})$ for its projections.

- Then we can define an endofunctor $\mathbb{Q}_{\mathscr{A}}$ as follows:

1. V($\mathbb{Q}_{\mathscr{A}}G$) is the set of all functions of the form $\psi : V(G) \to proj(\mathscr{A})$ where $\Sigma_{g \in V(G)} \psi(g) = I$.

2. $\psi \sim \psi'$ in $\mathbb{Q}_{\mathscr{A}}G$ if whenever $g \nsim g'$ in G then $\psi(g)\,.\,\psi(g') = 0$.

Putting a graded monad structure on $\mathbb{Q}_{\mathscr{A}}$?

- Can we use $\mathbb{Q}_{\mathscr{A}}$ to construct a graded quantum commuting monad?Coming up with a unit seems straightforward. $\eta_X(x) = I_1$

- Putting a multiplication structure on $\mathbb{Q}_{\mathscr{A}}$ is more difficult. it is not immediately clear what the grading should be.

- Consider the monoid $(M, \otimes, \mathbb{C})$ whose elements are unital C*-algebras. We can use this monoid to define a graded quantum commuting monad.

- Row quantum permutation matrices will then be composed in the kleisli category using the "Woronowicz tensor product":

$$u \oplus v := \sum_{i,j=1}^{N} E_{ij} \otimes \left(\sum_{k=1}^{N} u_{ik} \otimes v_{kj} \right).$$

Example: Quantum Commuting Monad

The (graded) **quantum commuting monad** $(\mathbb{Q}_{\mathscr{A}}, \eta, \mu^{\mathscr{A},\mathscr{B}})$ is given by:

1. $\mathbb{Q}_{\mathscr{A}}G$ is defined as before.
2. $\eta_X(x) = \delta_x \in \mathscr{Q}_{\mathbb{C}}X$ where for $x \neq x'$: $\begin{cases} \delta_a(a) = I_1 \\ \delta_a(a') = \mathbf{0}_1 \end{cases}$
3. $\mu_X^{\mathscr{A},\mathscr{B}}(\psi)(x) = \Sigma_{\phi \in \mathscr{Q}_{\mathscr{B}}X} \psi(\phi) \otimes \phi(x)$

Note that if we limit ourselves to those C*-algebras whose elements are n by n matrices we recover the quantum tensor monad.

1. $G \overset{co}{\to} H$ iff there exists a kleisli morphism $G \to \mathbb{Q}_{\mathscr{A}}H$
2. $G \cong H$ iff there exists a kleisli isomorphism $G \cong_{kl(\mathbb{Q}_{\mathscr{A}})} H$
3. $G \overset{co}{\cong} H$ iff $G \rightleftarrows_{kl(\mathbb{Q}_{\mathscr{A}})} H$

Non-Signalling strategies

- We define a variant of the distribution monad $\mathbb{D}$ on the category of graphs.

1. It acts exactly the same way as $\mathscr{D}$ on the set of vertices of the graph.

2. $\psi \sim \psi'$ if whenever $g \nsim g'$ in G then $\psi(g) \,.\, \psi(g') = 0$.

1. $G \stackrel{ns}{\rightarrow} H$ iff there exists a kleisli morphism $G \rightarrow \mathbb{D}H$
2. $G \cong H$ iff there exists a kleisli isomorphism $G \cong_{kl(\mathbb{D})} H$
3. $G \stackrel{ns}{\cong} H$ iff $G \rightleftarrows_{kl(\mathbb{D})} H$

- It is worth noting that non-signalling homomorphism is a trivial relationship. The players can perfectly win this game on almost any pair of graphs.

- Non-signalling isomorphism however, corresponds to fractional isomorphism, a well-studied linear algebraic relaxation of graph isomorphism

Conclusions

- We have provided a monadic account of perfect strategies for the graph homomorphism and isomorphism games.

- For each type of strategy $t \in \{c, *, co, ns\}$ the existence of a perfect strategy in the homomorphism game corresponds to the existence of a morphism in the kleisli category of a suitable (graded) monad.

- perfect strategies for the isomorphism game correspond to the existence of a suitable pair of back-and-forth morphisms in the same kleisli category. It remains to be seen if this description can be further refined.

- These ideas should be applicable to the well-studied class of synchronous non-local games, and to a generalisation of this class known as imitation games.