Concrete, Abstract, Axiomatic:

A (personal) Journey Through Game Comonads

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Game comonads

The Pebbling Comonad in Finite Model Theory

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***Abstract*—Pebble games are a powerful tool in the study of finite model theory, constraint satisfaction and database theory. Monads and comonads are basic notions of category theory which are widely used in semantics of computation and in modern functional programming. We show that existential k-pebble games have a natural comonadic formulation. Winning strategies for Duplicator in the k-pebble game for structures A and B are equivalent to morphisms from A to B in the coKleisli category for this comonad. This leads on to comonadic characterisations of a number of central concepts in Finite Model Theory:**

- **Isomorphism in the co-Kleisli category characterises elementary equivalence in the k-variable logic with counting quantifiers.**
- **Symmetric games corresponding to equivalence in full k-variable logic are also characterized.**
- **The treewidth of a structure A is characterised in terms of its coalgebra number: the least k for which there is a coalgebra structure on A for the k-pebbling comonad.**
- **Co-Kleisli morphisms are used to characterize strong consistency, and to give an account of a Cai-Fürer-Immerman construction.**
- **The k-pebbling comonad is also used to give semantics to a novel modal operator.**

These results lay the basis for some new and promising connections between two areas within logic in computer science which have largely been disjoint: (1) finite and algorithmic model theory, and (2) semantics and categorical structures of computation.

1. Introduction

Homomorphisms play a fundamental rôle in finite model theory, constraint satisfaction and database theory. The existence of a homomorphism $A \to B$ is an equivalent formulation of the basic CSP problem [1], [2], [3]. There is an equivalence between the existence of a homomorphism, and the property that every existential positive sentence satisfied by A is also satisfied by B [2]. Such sentences correspond to (disjunctions of) *conjunctive queries*, which are fundamental in database theory [4], [5].

One of the key tools in studying these notions is that of existential k-pebble games [6]. Such a game, for structures A, B, proceeds by Spoiler placing one of his k pebbles on an element of the universe of A. Duplicator then places one of her pebbles on an element of B. If Duplicator is always able to move so that the partial mapping from A to B defined by sending a_i, the element in A carrying the i'th Spoiler pebble, to b_i, the corresponding element of B carrying the i'th Duplicator pebble, is a homomorphism on the induced substructures, then Duplicator has a winning strategy.

Proposition 1 ([6]). *The following are equivalent:*

- *Duplicator has a winning strategy in the existential k-pebble game.*
- *Every sentence of the existential positive k-variable fragment of first-order logic satisfied by A is also satisfied by B.*

Our aim in this paper is to study these notions from a novel perspective, using the notion of *comonad* from category theory. Monads and comonads are basic notions of category theory which are widely used in semantics of computation and in modern functional programming [7], [8], [9]. We show that existential k-pebble games have a natural comonadic formulation. Given a structure A over a relational signature σ, we shall introduce a new structure $\mathbb{T}_k A$ corresponding to Spoiler playing his part of an existential k-pebble game on A, with the potential codomain B left unspecified. The idea is that we can exactly recover the content of a Duplicator strategy in B by giving a homomorphism from $\mathbb{T}_k A$ to B. Thus the notion of *local approximation* built into the k-pebble game is internalised into the category of σ-structures and homomorphisms. Formally, this construction will be shown to give a *comonad* on this category, which guarantees a wealth of further structural properties. This leads to comonadic characterisations of a number of central concepts in Finite Model Theory.

In Section 2, we introduce the pebbling comonads $\mathbb{T}_k$, which are graded by the number of pebbles k, and characterize their coalgebras. $\mathbb{T}_k A$ is always infinite. In Section 3, we prove a no-go theorem, to rule out any finite version. In Section 4, we show that the question of whether a morphism $\mathbb{T}_k A \to B$ exists is equivalent to the existence

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- Pebble comonad
  (Abramsky, Dawar & Wang, 2017)
- Ehrenfeucht-Fraïssé and modal comonads
  (Abramsky & Shah, 2018)
- Hella comonad
  (Ó Conghaile & Dawar, 2021)
- Guarded comonad
  (Abramsky & Marsden, 2021)
- Pebble-relation comonad
  (Montacute & Shah, 2022)
- Hybrid comonad
  (Abramsky & Marsden, 2022)

From Concrete to Abstract

Moving from the concrete setting of games to the abstract one of game comonads, we can

- better identify recurring patterns and structure
- isolate the aspects that are specific to the context from the "context-free" ones

Homomorphism counting (after Lovász)

Several homomorphism counting results in finite model theory can be understood as

categorical envelope + combinatorial core

The combinatorial core can be typically understood as an equality elimination result. (Dawar, Jakl & LR, 2021)

The aim is *not* to replace all combinatorial and game theoretic arguments with categorical ones, but rather to find a fruitful combination of the two.

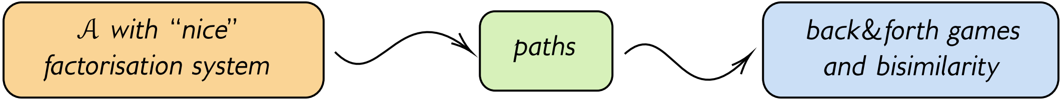
From Abstract to Axiomatic

A number of features appear to be common to all game comonads.

Can we reason about a generic game comonad?

... what is a *game comonad* ?

(Abramsky & LR, 2021) Arboreal categories as an axiomatic approach to game comonads:



The axioms for arboreal categories ensure that back-and-forth equivalence coincides with open-map bisimilarity in the sense of (Joyal, Nielsen & Winskel, 1993).

For example, axiomatic proofs of equi-resource Homomorphism Preservation Theorems can be established at the level of arboreal categories (Abramsky & LR).

Some Homotopical Aspects of Logic

The axiomatic language is powerful but, at the same time, points at some deficiencies in the way objects are manipulated.

In (finite) model theory, one is typically not interested in objects up to isomorphism, but only up to definable properties.

This is analogous to the idea from topology/homotopy theory that geometric objects should be studied only up to continuous deformation (homeomorphism/homotopy equivalence).

A homotopical approach to arboreal categories and resource-sensitive model theory would

- yield a flexible language to manipulate and construct objects "up to logical equivalence"
- clarify and simplify some important constructions in (finite) model theory, facilitating the transfer of methods and tools to other contexts.

Rossman's HPTs

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

The key idea is that of upgrading: given structures a, b, construct extensions a^*, b^* such that a^* and b^* are FO_k-equivalent whenever a and b are $\exists^+\mathrm{FO}_k$-equivalent.

$$\begin{array}{ccc} a^* & \leftrightarrow^k & b^* \\ \uparrow & & \uparrow \\ a & \leftrightarrows^k & b \end{array}$$

The proof of Rossman's Finite HPT follows a similar idea: given finite a, b, construct finite extensions a^*, b^* such that $a \leftrightarrows^\ell b$ entails $a^* \leftrightarrow^k b^*$ for ℓ sufficiently large.

In the axiomatic setting, the Equirank HPT can be proved by constructing the extension a^* starting from a, taking a (wide) pushout, and iterating the process ω times.

This is akin to a small object argument in (categorical) homotopy theory.

Upgrading

Upgrading arguments as the previous one are pervasive in (finite) model theory, see e.g.

Otto, *Model-theoretic methods for fragments of FO and special classes of (finite) structures.*

The general idea is that of constructing an "extension" of a given structure that preserves certain prescribed properties and, in addition, is symmetrical or "saturated". Cf. e.g. the construction of saturated elementary extensions in classical model theory. From the viewpoint of homotopy theory, this can be thought of as a form of fibrant replacement.

Can we make this analogy precise?

A (naive?) attempt: in the axiomatic setting, equivalence in various logic fragments is captured by spans of open morphisms. Is there a "homotopy structure" in which

open morphisms = weak equivalences ?

No!

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Weak Factorizations, Fractions and Homotopies

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Abstract. We show that the homotopy category can be assigned to any category equipped with a weak factorization system. A classical example of this construction is the stable category of modules. We discuss a connection with the open map approach to bisimulations proposed by Joyal, Nielsen and Winskel.

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Key words: weak factorization system, homotopy, category of fractions, bisimilarity.

1. Introduction

Weak factorization systems originated in homotopy theory (see [12, 7, 4, 3]). Having a weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$, we can formally invert the morphisms from $\mathcal{R}$ and form the category of fractions $\mathcal{K}[\mathcal{R}^{-1}]$. From the point of view of homotopy theory, we invert too few morphisms: only trivial fibrations and not all weak equivalences. Our aim is to show that this procedure is important in many situations.

For instance, the class *Mono* of all monomorphisms form a left part of the weak factorization system $(\mathit{Mono}, \mathcal{R})$ in a category R-**Mod** of (left) modules over a ring R. Then R-**Mod** $[\mathcal{R}^{-1}]$ is the usual stable category of modules. Or, in the open map approach to bisimulations suggested in [10], one considers a weak factorization system $(\mathcal{L}, \mathcal{O}_{\mathcal{P}})$, where $\mathcal{O}_{\mathcal{P}}$ is the class of $\mathcal{P}$-open morphisms w.r.t. a given full subcategory $\mathcal{P}$ of path objects. Then two objects K and L are $\mathcal{P}$-bisimilar iff there is a span

$$K \xleftarrow{f} M \xrightarrow{g} L$$

of $\mathcal{P}$-open morphisms. Any two $\mathcal{P}$-bisimilar objects are isomorphic in the fraction category $\mathcal{K}[\mathcal{O}_{\mathcal{P}}^{-1}]$ but, in general, the fraction category makes more objects isomorphic than just $\mathcal{P}$-bisimilar ones.

Any weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ with finite coproducts yields a cylinder object in $\mathcal{K}$ and thus a relation $\sim$ of homotopy between

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End of the story? Not quite...

Quillen model categories

A model category is a category $\mathcal{X}$ equipped with three classes of morphisms

$$\begin{aligned} \mathcal{W} &: \text{weak equivalences} \\ \mathcal{F} &: \text{fibrations} \\ \mathcal{C} &: \text{cofibrations} \end{aligned}$$

satisfying the following properties:

1. $\mathcal{X}$ is (finitely) complete and cocomplete;
2. $\mathcal{W}$ has the two-out-of-three property;
3. The pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorisation systems on $\mathcal{X}$.

For the duality theorists in the room

$$\text{sets} : \text{categories} = \text{topological spaces} : (\infty, 1)\text{-categories}$$

Quillen model categories are presentations of $(\infty, 1)$-categories.

Joyal's Proposition E.1.10

A model structure is determined by its cofibrations together with its class of fibrant objects.

A model category for modal logic

Let $\mathbf{K}$ be the category of Kripke models. For each $n \leq \omega$, the tree unravelling up to level n gives a coreflection

$$\mathbf{S}_n \underset{\longleftarrow\, R_n}{\overset{\top}{\hookrightarrow}} \mathbf{K}$$

of $\mathbf{K}$ into the full subcategory $\mathbf{S}_n$ consisting of synchronization trees of height at most n.
For all $a \in \mathbf{K}$, a and $R_n a$ satisfy the same modal formulas with modal depth $\leq n$.

- $\mathbf{P}_n$: subcategory of $\mathbf{S}_n$ whose objects are the synchronization trees with a single branch, i.e. the traces, and whose morphisms are the embeddings.
- $\mathbf{S}_n^*$: wide subcategory of $\mathbf{S}_n$ whose morphisms (preserve and) reflect the unary relations. The restricted Yoneda embedding gives $\mathbf{S}_n^* \hookrightarrow \widehat{\mathbf{P}_n}$.

The presheaf category $\widehat{\mathbf{P}_n}$ admits a Quillen model structure such that any $X \xrightarrow{f} Y$ in $\mathbf{S}_n^*$

- is an embedding just when it is a cofibration.
- is a p-morphism (i.e., bounded) just when it is a trivial fibration.

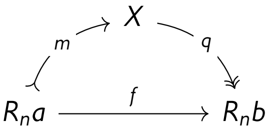
A preservation theorem for modal logic

Theorem (Andréka, van Benthem & Németi; Rosen)

A modal formula of depth $\leq n$ is preserved under embeddings of Kripke models iff it is equivalent to an existential modal formula of depth $\leq n$ (constructed from the atoms and their negations, using $\wedge, \vee$ and $\Diamond$). Further, the result relativises to finite Kripke models.

Proof. Let φ be a modal formula of depth n preserved under embeddings, and let $a, b \in \mathbf{K}$ satisfy $a \Rrightarrow_n^{\exists} b$. We must prove that $a \models \varphi \; \Rightarrow \; b \models \varphi$. Equivalently, $R_n a \models \varphi \; \Rightarrow \; R_n b \models \varphi$.

The condition $a \Rrightarrow_n^{\exists} b$ implies the existence of a morphism $f \colon R_n a \to R_n b$ in $\mathbf{S}_n^*$. Take the $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ factorisation of f in $\widehat{\mathbf{P}}_n$:



One proves that $X \in \mathbf{S}_n^*$, and so m is an embedding and q is a p-morphism. Thus, $R_n a \models \varphi \; \Rightarrow \; R_n b \models \varphi$.

Further, if a, b are finite, so is X. $\square$

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