A Synthetic Road to Locality Theorems

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18 July 2023

Resources and Co-Resources Workshop, Cambridge

The categorical story so far

Games vs comonads

Basic intuition:

(well-behaved) model comparison games $\longleftrightarrow$ comonads

More concretely:

one-way games for $\Rrightarrow_{\exists^+\mathbb{C}}$	$\longleftrightarrow$	morphisms in $\mathsf{Kl}(\mathbb{C})$
bijective games for $\equiv_{\#\mathbb{C}}$	$\longleftrightarrow$	isomorphism in $\mathsf{Kl}(\mathbb{C})$
back-and-forth games for $\equiv_{\mathbb{C}}$	$\longleftrightarrow$	bisimulation in $\mathsf{EM}(\mathbb{C})$

Characterisations of existential fragments and positive fragments exist too.

Combinatorial properties

Basic intuition:

(well-behaved) decompositions	$\longleftrightarrow$	comonad coalgebras
(well-behaved) unstructured combinatorial properties	$\longleftrightarrow$	weakly initial comonads

[AJP'22]: Any property Δ of graphs/structures such that $A + B \in \Delta$ iff $A, B \in \Delta$ is classified by a weakly initial comonad.

Lovász-type Counting theorems

Basic intuition:

isomorphism from homomorphism counting	$\longleftrightarrow$	combinatorial categories
log. equivalence $\equiv_{\#\mathbb{C}}$ from homomorphism counting	$\longleftrightarrow$	combinatoriality for (finite) coalgebras of $\mathbb{C}$

[DJR'21]: combinatoriality for coalgebras $\leftarrow$ (co)monadicity for comonads that preserve finiteness

[Reggio'22]: combinatoriality for coalgebras $\leftarrow$ (co)monadicity + lfp categories for comonads of finite rank

[AJP'22]: makes use of [Reggio'22] for weakly initial comonads

Transformations, i

decomposition transformations $\longleftrightarrow$ Eilenberg–Moore lifts H^{EM}

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{H^{\mathsf{EM}}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array}$$

$\longleftrightarrow$ Eilenberg–Moore laws $H\mathbb{C} \Rightarrow \mathbb{D}H$

$\Rrightarrow_{\exists^+\mathbb{C}}$ and $\equiv_{\#\mathbb{C}}$ transformations $\longleftrightarrow$ Kleisli lifts H^{Kl}

$$\begin{array}{ccc} \mathsf{Kl}(\mathbb{C}) & \xrightarrow{H^{\mathsf{Kl}}} & \mathsf{Kl}(\mathbb{D}) \\ {\scriptstyle F}\big\uparrow & & \big\uparrow{\scriptstyle F} \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array}$$

$\longleftrightarrow$ Kleisli laws $\mathbb{D}H \Rightarrow H\mathbb{C}$

Transformations, ii

If $\mathsf{EM}(\mathbb{D})$ has suitable equalisers[1]:

given by $\mathbb{D}H \Rightarrow H\mathbb{C}$

$\equiv_{\mathbb{C}}$ transformations $\longleftrightarrow$ "**full**" Kleisli lifts

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D}) \\ \uparrow F & & \uparrow F \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array}$$

$+$ axioms (S1), (S2) for $\widehat{H}$

(S1) $\widehat{H}$ preserves embeddings

(S2) path embeddings $P \rightarrowtail \widehat{H}X$ have a minimal decomposition via $\widehat{H}(e)$ of some $e\colon P' \rightarrowtail X$

[1] Usually enough to check that $\mathbb{D}$ preserves embeddings.

Our minimal synthetic setup

Elementary path categories

(Inspired by the arboreal approach of Luca and Samson [AR'21].)

An **elementary path category**[2] is a triple $(\mathscr{X}, \mathscr{M}, \mathscr{P})$ where

- $\mathscr{M}$ is a collection of **embeddings** $\rightarrowtail$, i.e. morphisms in $\mathscr{X}$ s.t.
  1. $\mathscr{M} \subseteq \{\text{monos}\}$
  2. $f, g \in \mathscr{M}$ implies $fg \in \mathscr{M}$ (if defined)
  3. $fg \in \mathscr{M}$ implies $g \in \mathscr{M}$
- $\mathscr{P}$ is a set of **paths**, i.e. objects in $\mathscr{X}$

For simplicity: for paths $P \cong Q \implies P = Q$

We add further axioms as needed (usually inspired by arboreal theorems).

[2] I can't decide on the name: elementary path, ramus, prearboreal, ...?

Elementary path adjunctions

Typical situation

$$\begin{array}{c} \mathscr{X} \\ U \Big\downarrow \; \dashv \; \Big\uparrow F \\ \mathscr{A} \end{array}$$

from $\mathbb{C} = \mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k, \ldots$

$$\begin{array}{c} \mathsf{EM}(\mathbb{C}) \\ U \Big\downarrow \; \dashv \; \Big\uparrow F \\ \mathcal{R}(\sigma) \end{array}$$

Usually $\mathscr{A}$ equipped with embeddings, both U, F preserve these.

Detour: a new no-go theorem

Composition methods for products, i

For any comonad C on $\mathscr{A}$ with products, we have a Kleisli law

$$\mathbb{C}(A \times B) \to \mathbb{C}A \times \mathbb{C}B$$

Corollary

- $A \Rrightarrow_{\exists^+\mathbb{C}} B$ *and* $A' \Rrightarrow_{\exists^+\mathbb{C}} B'$ *implies* $A \times A' \Rrightarrow_{\exists^+\mathbb{C}} B \times B'$
- $A \equiv_{\#\mathbb{C}} B$ *and* $A' \equiv_{\#\mathbb{C}} B'$ *implies* $A \times A' \equiv_{\#\mathbb{C}} B \times B'$

ref: [JMS'23]

Composition methods for products, ii

Theorem

For a comonad $\mathbb{C}$ on a category with coproducts and a well-powered proper factorisation system such that

- $\mathbb{C}$ *preserves embeddings*
- *paths in* $\mathsf{EM}(\mathbb{C})$ *are closed under quotients*

We obtain

- $A \equiv_{\mathbb{C}} B$ *and* $A' \equiv_{\mathbb{C}} B'$ *implies* $A \times A' \equiv_{\mathbb{C}} B \times B'$

Holds for any $\mathbb{C}$ such that $\mathsf{EM}(\mathbb{C})$ is an arboreal category or even elementary path category!

ref: [JMS'23]

~~No-go theorem for fixpoints~~ (THIS SLIDE WAS WRONG)

Alexander Rabinovich:
On Compositionality and Its Limitations

shows that the UNTIL and EG modalities are incompatible with ~~product~~ product-like composition theorems.

$\Rightarrow$ ~~expressing logics by bisimulation in EM($\mathbb{C}$) for some comonad $\mathbb{C}$ such that EM($\mathbb{C}$) is arboreal or elementary path is impossible!~~

~~Alternatively, EM($\mathbb{C}$) could have has paths not closed under quotients.~~

$\Rightarrow$ ~~our usual approach fails for LTL, CTL, μ-calculus, ... but these logics admit game theoretic characterisations!~~

Locality theorems

!! WARNING !!

Work in progress ahead

Hanf locality with thresholds

For $a \in A$, define

$$\mathcal{N}_r(a) = \{x \in A \mid \delta(a, x) \leq d\}.$$

For an isomorphism r-type τ, define

$$\#\tau\langle A\rangle = \{a \in A \mid (\mathcal{N}_r(a), a) \cong \tau\}.$$

Theorem (Fagin–Stockmeyer–Vardi, 1995)

$\forall k, f \;\; \exists\, r, t$ such that, for graphs A and B with neighbourhoods of size $\leq f$,

$A \equiv_k B$ if $\forall$isomorphism r-type τ, either

- $\#\tau\langle A\rangle \cong \#\tau\langle B\rangle$ *or*
- *both $\#\tau\langle A\rangle$ and $\#\tau\langle B\rangle$ are at least t.*

Gaifman locality

Theorem (Gaifman, 1982)

Every first-order sentence is equivalent to a Boolean combination of **basic local sequences**, *that is, sentences of the form*

$$\exists \overline{x} \; (\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \; \wedge \; \bigwedge_{i} \theta(x_i))$$

where θ *is* r-**local**, *i.e.* $A \models \theta(a)$ *iff* $\mathcal{N}_r(a) \models \theta(a)$.

Theorem (Gaifman locality with thresholds)

For structures A, B,

$$A \cong^{r(k)}_{q(k)} B \quad \textit{implies} \quad A \equiv_k B$$

where $\cong^r_q$ *expresses equivalence w.r.t. basic local sentences of radius* r *and quantifier rank* q.

Proof structure of Hanf and Gaifman

Fix suitable radii $r_1, \ldots, r_k$ and quantifier ranks $q_1, \ldots, q_k$.

Invariant for position $\overline{a}, \overline{b}$ at round m

$$\bigcup_{i=1}^{m} \mathcal{N}_{r_m}(a_i) \quad \equiv_{q_m} \quad \bigcup_{i=1}^{m} \mathcal{N}_{r_m}(b_i) \qquad (\mathrm{inv}_m)$$

resp. $\cong$ for Hanf

Given $a \in A$, two cases:

1. $\mathcal{N}_{r_{m+1}}(a) \subseteq \bigcup_{i=1}^{m} \mathcal{N}_{r_m}(a_i)$

   use (inv_m) to find $b \in B$ such that $\overline{a}a, \overline{b}b$ satisfy (inv_{m+1})

2. $\mathcal{N}_{r_{m+1}}(a) \not\subseteq \bigcup_{i=1}^{m} \mathcal{N}_{r_m}(a_i)$

   • use (inv_m) for a bijection between "suitable subsets"

   • by thm. assumption, find $b \in B$ with (1) $\mathtt{tp}(a) = \mathtt{tp}(b)$, and (2) $\forall i \ \mathcal{N}_{r_{m+1}}(b_i) \cap \mathcal{N}_{r_{m+1}}(b) = \emptyset \quad \Rightarrow (\mathrm{inv}_{m+1})$

Why locality theorems?

- Important tool in Finite Model Theory:
  - Algorithmic usage for FPT decidability results.
  - Inexpressibility results.
  - ... but the variants reproved over and over again.

- No account of locality in categorical logic yet.

- It helped to identify uniformly quasiwide/nowhere dense classes i.e. to go much beyond bounded tree-width!

  Equivalently viewed as “sparse neighbourhood covers” – looks a bit like indexed Grothendieck topology with bits of comonadic structure!

Locality comonad?

Tom Paine's in his thesis

- established that there is no comonad on $\mathbb{E}_k(\mathcal{N}_r(-))$
- defined a "reachability comonad" $\mathbb{R}_m$ for an invariant similar to (inv$_m$), and hints at $\mathbb{E}_k \Rightarrow \mathbb{R}_k$

We need a comonad $\mathbb{C}_m$ to express the assumptions, i.e. the relation $\cong_q^r$ or equivalence wrt $\#\tau\langle \cdot \rangle$

... but case 2 is very "non-uniform", there is a counting argument

... we do not expect $\mathbb{E}_k \Rightarrow \mathbb{C}_{m(k)}$ or $\mathbb{R}_k \Rightarrow \mathbb{C}_{m(k)}$ satisfying (S2)

Instead, we specify $\cong_q^r$ as an extra structure!

The synthetic method (taken from Andrej Bauer)

1. Take a classic theorem in ~~computability theory~~ Finite Model Theory.
2. Rephrase it as a fact about ~~the effective topos~~ concrete game comonads.
3. Find a statement (for elementary path categories) whose interpretation is the fact.
4. Abstract the statement to expose its essence.
5. Give a synthetic proof.

Do not skip any steps! This can hinder progress significantly!

Steps ahead

1. What are formulas?
2. What are local formulas?
3. What are basic local formulas?
4. State the theorem.
5. Give a copy-cat proof.
6. (Future work:) synthesise the statement and its proof.

Formulas with free variables

Classically, $\Delta \subseteq \mathcal{R}_n(\sigma)$ is a **formula** of quantifier rank $\leq k$ iff

$$(A, \overline{a}) \in \Delta \quad \text{and} \quad (A, \overline{a}) \equiv_k (B, \overline{b}) \quad \text{implies} \quad (B, \overline{b}) \in \Delta$$

Constants $\overline{a} = (a_1, \ldots, a_n) \in A^n$

$\cong$ assignments/function $\{1, \ldots, n\} \to A$

$\cong$ homomorphism from a discrete $\{1, \ldots, n\}$ to A

$\cong$ coalgebra homomorphisms $\mathbf{p}_n \to F^{\mathbb{E}_{k+n}}(A)$

where $\mathbf{p}_n$ is the discrete chain $(1 < \cdots < n)$ in $\mathrm{EM}(\mathbb{E}_{k+n})$

$\Rightarrow U^{\mathbb{E}_{k+n}}(\mathbf{p}_n) = \{1, \ldots, n\}$

Types

Question: Given $(A, \overline{a})$ and $(B, \overline{b})$ as

$$F^{\mathbb{E}_{k+n}}(A) \xleftarrow{\overline{a}} \mathbf{p}_n \xrightarrow{\overline{b}} F^{\mathbb{E}_{k+n}}(A) \ ,$$

how do we express $(A, \overline{a}) \equiv_k (B, \overline{b})$ in $\mathrm{EM}(\mathbb{E}_{k+n})$?

Define a **(weak) type** $\mathtt{tp}(x)$ of $x \colon P \to X$ in $\mathrm{EM}(\mathbb{E}_{k+n})$, where P is a path, as the upset of the image of x in X.

$\mathtt{tp}(x) = \mathrm{colim}\{e \colon Q \rightarrowtail X \mid x \text{ factors via } e\}$

Then x factors as

$$P \xrightarrow{x^{\uparrow}} \mathtt{tp}(x) \rightarrowtail X$$

Theorem

$(A, \overline{a}) \equiv_k (B, \overline{b})$ *iff* $\mathtt{tp}(\overline{a}) \sim \mathtt{tp}(\overline{b})$

Strong functors, i

What functors preserve equivalence of types?

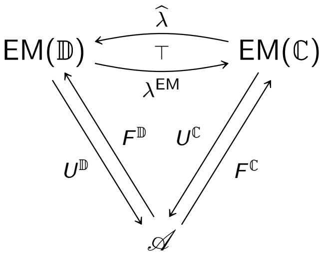
Our usual situation, a comonad morphism $\lambda\colon \mathbb{D} \Rightarrow \mathbb{C}$ yields

- λ^{EM} distributes over U's
- $\widehat{\lambda}$ distributes over F's
- $\widehat{\lambda}$ preserves embeddings
- λ mono $\Rightarrow \lambda^{\mathsf{EM}}$ fully faithful

♣ conjugation

$$\frac{f : X \to \widehat{\lambda}(Y)}{f^{\flat} \colon \lambda^{\mathsf{EM}}(X) \to Y}$$

often preserves path embeddings!



Strong functors, ii

Lemma (!! & ?!)

If λ^{EM} preserves paths and λ consists of embeddings then the conjugation preserves path embeddings. ($\Rightarrow \widehat{\lambda}$ *satisfies (S1), (S2))*

Lemma

For $L \dashv R$ between $\mathit{arboreal}$ categories. If conjugating preserves path embeddings and L is full then R preserves paths.

A functor $H\colon \mathscr{X} \longrightarrow \mathscr{Y}$ between elementary path categories is a **strong (path) functor** if

- H preserves embeddings
- H preserves paths
- has a left adjoint H_*
- the conjugation $f \mapsto f^\flat$ of $H_* \dashv H$ preserves path embeddings

(Weak functor would not preserve paths.)

Strong functors, iii

Theorem

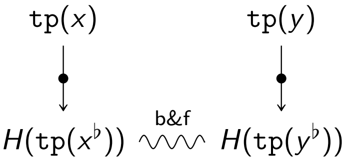
If $H\colon \mathscr{X} \longrightarrow \mathscr{Y}$ *is a strong path functor then, for*

$$x\colon P \to H(X) \quad \textit{and} \quad y\colon P \to H(Y) \qquad \textit{in } \mathscr{Y},$$

we have that

$$\mathtt{tp}(x^\flat) \sim \mathtt{tp}(y^\flat) \quad \textit{implies} \quad \mathtt{tp}(x) \sim \mathtt{tp}(y).$$

Proof idea.



□

Neighbourhood operators

Given any $(A, \overline{a})$ by

$$P \xrightarrow{x} F^{\mathbb{C}}(A)$$

we assume factorisation

$$U^{\mathbb{C}}(P) \longrightarrow N(x) \rightarrowtail A$$

giving us

$$P \xrightarrow{x^N} F^{\mathbb{C}}(N(x)) \rightarrowtail F^{\mathbb{C}}(A)$$

Remarks:

- usually $N(x) = \bigcup_{i=1}^{n} \mathcal{N}_r(x_i)$ for $x = (x_1, \ldots, x_n) \in A^n$
- usually a natural transformation $N \Rightarrow \mathrm{Id}$ on $U^{\mathbb{C}} \downarrow \mathscr{A}$

Local types and local formulas

Given any

$$P \xrightarrow{x} F^{\mathbb{C}}(A)$$

and a neighbourhood operator N for P, N-**local type** $\mathtt{ltp}_N(x)$ is the type of x^N in $F^{\mathbb{C}}(N(x))$, that is,

$$P \xrightarrow{(x^N)^{\uparrow}} \mathtt{ltp}_N(x) = \mathtt{tp}(x^N) \rightarrowtail F^{\mathbb{C}}(N(x))$$

For a collection $\Delta \subseteq \{x \colon P \to F^{\mathbb{C}}(A)\}_{A \in \mathscr{A}}$ (i.e. $\Delta \subseteq P \downarrow F^{\mathbb{C}}$)

Δ is a **formula** if $x \in \Delta$ and $\mathtt{tp}(x) \sim \mathtt{tp}(y)$ implies $y \in \Delta$

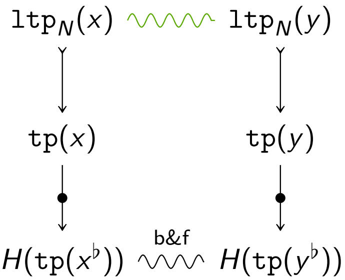
Δ is a **local formula** if $x \in \Delta$ and $\mathtt{ltp}(x) \sim \mathtt{ltp}(y)$ implies $y \in \Delta$

Detecting neighbourhoods

Theorem

If H **detects** *N, $\mathtt{tp}(x^\flat) \sim \mathtt{tp}(y^\flat)$ implies $\mathtt{ltp}_N(x) \sim \mathtt{ltp}_N(y)$.*

A strong path functor $H\colon \mathscr{X} \longrightarrow \mathscr{Y}$ which **detects** N allows to restrict bisimulation to N-local types:



Intuitively: $\mathscr{X}$ can express "$z \in N(x)$"

Basic local formulas

We want to mimic formulas $\exists \overline{x} \; (\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \;\wedge\; \bigwedge_i \theta(x_i))$

For an N-local formula Δ, a **basic local formula** is

$$\{A \mid \exists \text{ scattered } x_1, \ldots, x_n \colon P \to F^{\mathbb{C}}(A) \text{ in } \Delta\}$$

where **scattered** means $N(x_i) \rightarrowtail A \leftarrowtail N(x_j)$ are disjoint $\forall i \neq j$.

Equivalently, the induced $F^{\mathbb{C}}(N(x_1)) \oplus \cdots \oplus F^{\mathbb{C}}(N(x_n)) \rightarrowtail F^{\mathbb{C}}(A)$ is a pathwise embedding.

These are formulas, relative to a strong $H \colon \mathscr{X} \to \mathscr{Y}$ detecting N, assuming

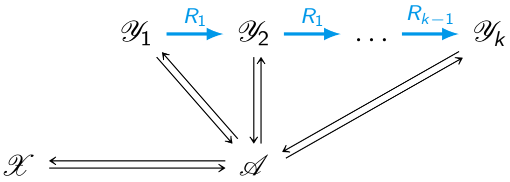
- a monoidal structure $\oplus$, preserved by H_*
- decomposition of discrete paths $\mathbf{p}_n = \mathbf{t}_1 \oplus \cdots \oplus \mathbf{t}_n$

e.g. $\mathscr{Y} = \mathrm{EM}(\mathbb{E}_k^{\text{⏲}})$ where **timed** $\mathbb{E}_k^{\text{⏲}}(A)$ consists of $[(m_1, a_1), \ldots, (m_n, a_n)]$ with $1 \leq m_1 < \cdots < m_n \leq k$

The statement structure $\rightsquigarrow$ step 3 ✓

<u>Original claim:</u> $A \cong^{r(k)}_{q(k)} B$ implies $A \equiv_k B$

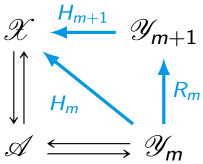
Instead of $r_1, \ldots, r_k$ and $q_1, \ldots, q_k$ we fix:



We want $F^{\mathscr{X}}(A) \sim F^{\mathscr{X}}(B)$. From

- "discrete" paths $\mathbf{p}_1 \in \mathscr{Y}_1, \ldots, \mathbf{p}_k \in \mathscr{Y}_k$
  where $\mathbf{p}_{i+1}$ extends $R_i(\mathbf{p}_i)$
- neighbourhood operators $N_1, \ldots, N_k$ for $\mathbf{p}_1, \ldots, \mathbf{p}_k$

The proof (inductive step)



Invariant (inv_m)

$$\bigcup_{i=1}^{m} \mathcal{N}_{r_m}(a_i) \equiv_{q_m} \bigcup_{i=1}^{m} \mathcal{N}_{r_m}(b_i)$$

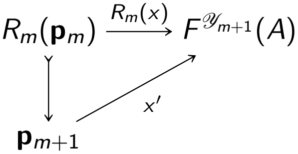
replaced by

$$\mathtt{ltp}_{N_m}(x) \sim \mathtt{ltp}_{N_m}(y) \text{ in } \mathscr{Y}_m$$

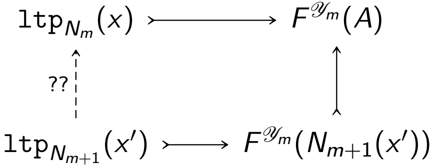
for assignments

$$x\colon \mathbf{p}_m \to F^{\mathscr{Y}_m}(A) \qquad y\colon \mathbf{p}_m \to F^{\mathscr{Y}_m}(B)$$

Next step – extension:



Gaifman's two cases:



Thank you!

Added axioms

For an elementary path category $\mathscr{X}$, we needed to further assume:

- all $P \to X$ have a minimal decomposition $P \to P' \rightarrowtail X$
- morphisms $P \to \operatorname{colim} \mathscr{D}$, where $\mathscr{D}$ is a diagram of paths and embeddings, factor through one of the inclusions $d \to \operatorname{colim} \mathscr{D}$
- for a full downset subcategory $\mathscr{D}$ of $\mathit{Paths}(X)$, the colimit of $\mathscr{D}$ exists and the induced $\operatorname{colim} \mathscr{D} \to X$ is an embedding

Finally, for our adjunction $U \dashv F$ between $\mathscr{A} \leftrightarrows \mathscr{X}$,

- $\mathscr{A}$ also has embeddings and F (and sometimes also U) is required to preserve them