Sweedler theory of monads

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What is this about?

- Structure, I am afraid...

- We are following Moggi's monad-based approach to effects in mathematical semantics of functional programming.

- Effects are a program's (a computation's) requests to the outside world for certain services.
- To be able to run, a program has to meet a state machine (an environment) able to serve these requests.
- The two have to understand each other.

- Monad-comonad interaction laws mathematize the communication protocols between computations and environments.

- How to find the universal notion of environment for the given notion of computation and vice versa?

Outline

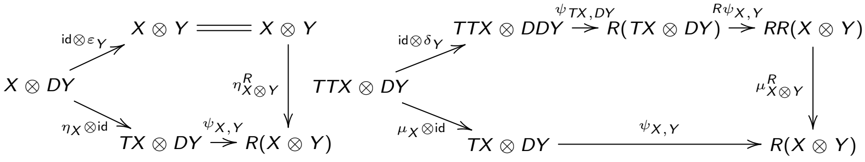
- Monad-comonad interaction laws (Katsumata, R., U.)
- An abstract (Sweedler theory) view:
  measuring maps in duoidal Sweedler theory
- A (co)algebraic characterization (U., Voorneveld)
- Combining the Sweedler theory and (co)algebraic perspectives

Monad-comonad interaction laws

- Let $\mathbb{C}$ be a symm. mon. category.
- A *monad-comonad interaction law* is a monad (T, η, μ), a comonad (D, ε, δ) and monad (R, η^R, μ^R) and a nat. transf. typed

$$\psi_{X,Y} : TX \otimes DY \to R(X \otimes Y)$$

  such that



- Legend:
  T – notion of computation, X – values
  D – notion of environment, Y – states
  R – notion of residual computation
- The most important case is $R = \mathrm{Id}$.

Example: State (1)

- Let $\mathbb{C}$ be a CCC, e.g., Set.

- $TX = S \Rightarrow (S \times X)$ (the state monad)
- $DY = S \times (S \Rightarrow Y)$ (the costate comonad)
  for some S
- $RZ = Z$

- $\psi : (S \Rightarrow (S \times X)) \times (S \times (S \Rightarrow Y)) \to X \times Y$
  $\psi\,(f, (s, g)) = \text{let } (s', x) = f\ s \text{ in } (x, g\ s')$

- Legend:
  X – values
  Y – (control) states, S – stores (data states)

Example: State (2)

- $TX = V \Rightarrow (V \times X)$ (the state monad)
- $DY = S \times (S \Rightarrow Y)$ (the costate comonad)
  for some S, V, $get : S \to V$ and $put : S \times V \to S$
  forming a *(very well-behaved) lens*
- $RZ = Z$
- $\psi : (V \Rightarrow (V \times X)) \times (S \times (S \Rightarrow Y)) \to X \times Y$
  $\psi\,(f,(s,g)) = \text{let } (v',x) = f\,(get\,s) \text{ in } (x, g\,(put\,(s,v')))$
- Legend:
  X – values, V – "views" of stores (data states),
  Y – (control) states, S – stores (data states)

Example: State (3)

- $TX = \mu X'. X + (S \Rightarrow X') + (S \times X')$
  (the intensional state monad)
- $DY = S \times (S \Rightarrow Y)$
- $RZ = Z$

- $\psi\,(\mathsf{inl}\,x, (s, g)) = (x, g\,s)$
  $\psi\,(\mathsf{inr}\,(\mathsf{inl}\,f), (s, g)) = \psi\,(f\,s, (s, g))$
  $\psi\,(\mathsf{inr}\,(\mathsf{inr}\,(s', c)), (\_, g)) = \psi\,(c, (s', g))$

- $TX = S \Rightarrow X$ (the reader monad)
- $DY = S \times (S \Rightarrow Y)$
- $RZ = Z$

- $\psi\,(f, (s, g)) = (f\,s, g\,s)$

Example: (Intensional) nondeterminism

- $TX = \mu X'.\, X + X' \times X'$
- $DY = \nu Y'.\, Y \times (Y' + Y') \cong \nu Y'.\, Y \times (2 \times Y') \cong \mathsf{Str}\,(Y \times 2)$
- $RZ = Z$

- $\psi\,(\mathsf{inl}\, x, (y, \_)) = (x, y)$
  $\psi\,(\mathsf{inr}\,(c, \_), (\_, \mathsf{inl}\, e) = \psi\,(c, e)$
  $\psi\,(\mathsf{inr}\,(\_, c), (\_, \mathsf{inr}\, e) = \psi\,(c, e)$

- $TX = \mu X'.\, X + (1 + X' \times X')$
- $DY = \nu Y'.\, Y \times (Y' + Y')$
- $RZ = Z + 1$

- $\psi\,(\mathsf{inl}\, x, (y, \_)) = \mathsf{inl}\,(x, y)$
  $\psi\,(\mathsf{inr}\,(\mathsf{inl}\,\star)), \_) = \mathsf{inr}\,\star$
  $\psi\,(\mathsf{inr}\,(\mathsf{inr}\,(c, \_)), (\_, \mathsf{inl}\, e) = \psi\,(c, e)$
  $\psi\,(\mathsf{inr}\,(\mathsf{inr}\,(\_, c)), (\_, \mathsf{inr}\, e) = \psi\,(c, e)$

Example: Nontermination

- $TX = \nu X'. X + X'$ (the delay monad)
- $DY = \mu Y'. Y \times (1 + Y') \cong \mathsf{NEList}\, Y$ (the timeout comonad)
- $RZ = TZ$

- $\psi\,(\mathsf{inl}\, x, (y, \_)) = \mathsf{inl}\,(x, y)$
  $\psi\,(\mathsf{inr}\, c, (y, \mathsf{inl}\, \star) = \mathsf{inr}\,(\psi\,(c, (y, \mathsf{inl}\, \star)))$
  $\psi\,(\mathsf{inr}\, c, (\_, \mathsf{inr}\, e) = \psi\,(c, e)$

Alternative formulations

- If $\mathbb{C}$ is closed,
  the definition of mnd.-cmnd. int. laws admits further variants:

$$
\begin{array}{c}
TX \otimes DY \to R(X \otimes Y) \text{ nat. in } X, Y \text{ subj. to eqs.} \\
\hline\hline
\mathbb{C}(X \otimes Y, Z) \to \mathbb{C}(TX \otimes DY, RZ) \text{ nat. in } X, Y, Z \text{ subj. to eqs.} \\
\hline\hline
T(Y \multimap Z) \to DY \multimap RZ \text{ nat. in } Y, Z \text{ subj. to eqs.} \\
\hline\hline
D(X \multimap Z) \to TX \multimap RZ \text{ nat. in } X, Z \text{ subj. to eqs.}
\end{array}
$$

- Legend:
  X – values
  Y – states
  Z – observables (values for residual computations)
  $X \otimes Y \to Z$ – observation functions

Monad-comonad interaction laws as monoids

- A *functor-functor interaction law* is given by three functors $F, G, H : \mathbb{C} \to \mathbb{C}$ and a nat. transf. typed maps

$$\phi_{X,Y} : FX \otimes GY \to H(X \otimes Y)$$

- A *functor-functor interaction law map* between (F, G, H, ϕ), (F', G', H', ϕ') is given by nat. transfs. $f : F \to F'$, $g : G' \to G$, $h : H \to H'$ such that

$$\begin{array}{ccccc} & & FX \otimes GY & \xrightarrow{\phi_{X,Y}} & H(X \otimes Y) \\ & \overset{\mathrm{id} \otimes g_Y}{\nearrow} & & & \downarrow{\scriptstyle h_{X \otimes Y}} \\ FX \otimes G'Y & & & & \\ & \underset{f_X \otimes \mathrm{id}}{\searrow} & & & \\ & & F'X \otimes G'Y & \xrightarrow{\phi'_{X,Y}} & H'(X \otimes Y) \end{array}$$

- Functor-functor int. laws form a category with a composition-based monoidal structure.
- Monad-comonad int. laws are monoids in this category.

R-residual monad-comonad interaction laws as monoids

- One can fix H to be the underlying functor R of some particular monad (R, η^R, μ^R).

- The category of R-residual functor-functor int. laws has a composition-based monoidal structure using (η^R, μ^R).
- R-residual monad-comonad int. laws are monoids in this category.

Degeneracies for $R = \mathrm{Id}$

- Assume $\mathbb{C}$ is extensive.
- If F comes with a nullary operation or a commutative binary operation and interacts with G, then $GY \cong 0$.
- If T comes with an associative binary operation and interacts with D, then D cannot be very interesting.
- It is therefore often useful to use, e.g., $- + 1$, $\mathcal{M}_{\mathrm{f}}^{+}$ or $\mathcal{M}_{\mathrm{f}}$ as the monad R.

A challenge

- We would like to be able, given a monad R,
  - to construct the final monad T interacting R-residually with a given comonad D,
  - or to construct the final comonad D interacting R-residually with a given monad T,
- or also, given a monad T and a comonad D,
  - to construct the initial monad R wrt. which they can interact residually;
- in short, given any two,
  - to construct the universal third.

Abstracting to monoid-comonoid interaction laws

- Assume the symm. mon. cat $\mathbb{C}$ is locally presentable. Cut $[\mathbb{C}, \mathbb{C}]$ down to accessible functors.
- $[\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ has a Day convolution symm. mon. structure.

$$JZ = \mathbb{C}(\mathsf{I}, Z) \bullet \mathsf{I}$$
$$(F \star G)Z = \int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (FX \otimes GY)$$

- Func.-func. int. laws for F, G, H are in bijection with maps $F \star G \to H$.

$$\frac{\dfrac{FX \otimes GY \to H(X \otimes Y) \text{ nat. in } X, Y}{\mathbb{C}(X \otimes Y, Z) \to \mathbb{C}(FX \otimes GY, HZ) \text{ nat. in } X, Y, Z}}{\underbrace{\int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (FX \otimes GY)}_{(F \star G)Z} \to HZ \text{ nat. in } Z}$$

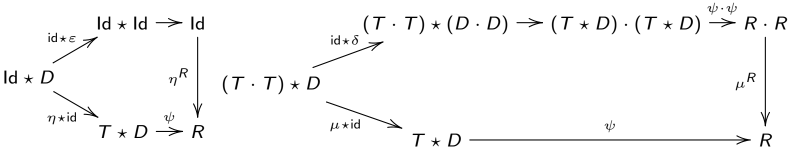
- The category of R-residual func.-func. interaction laws is isomorphic to that of Chu spaces with vertex R.

Abstracting to monoid-comonoid interaction laws ctd.

- Composition and Day convolution together equip $[\mathbb{C}, \mathbb{C}]_a$ with a *duoidal* structure $(\mathsf{Id}, \cdot, J, \star)$.
- In particular, $\star$ is oplax monoidal wrt. $(\mathsf{Id}, \cdot)$, so there are structural laws

$$\mathsf{Id} \star \mathsf{Id} \to \mathsf{Id}$$
$$(F \cdot F') \star (G \cdot G') \to (F \star G) \cdot (F' \star G')$$

  subject to the right equations.
- Mnd.-cmnd. int. laws for T, D, R are in bijection with maps $T \star D \to R$ such that



- We can abstract from $[\mathbb{C}, \mathbb{C}]_a$ and talk about object-object and monoid-comonoid int. laws in a general symm. duoidal category.

Abstracting to monoid-comonoid interaction laws ctd.

- If $\mathbb{C}$ is closed, i.e., $- \otimes Y$ has a right adjoint $Y \multimap -$, then $- \star G$ has a right adjoint $G \rightarrow\!\!\star\, -$ given by

$$(G \rightarrow\!\!\star\, H)X = \int_Y GY \multimap H(X \otimes Y)$$

- For a comonad (D, ε, δ) and a monad (R, η^R, μ^R), the functor $D \rightarrow\!\!\star\, R$ is a monad via

$$\eta = \ \mathsf{Id} \longrightarrow \mathsf{Id} \rightarrow\!\!\star\, \mathsf{Id} \xrightarrow{\varepsilon \rightarrow\!\star\, \eta^R} D \rightarrow\!\!\star\, R$$

$$\mu = \ (D \rightarrow\!\!\star\, R) \cdot (D \rightarrow\!\!\star\, R) \longrightarrow (D \cdot D) \rightarrow\!\!\star\, (R \cdot R) \xrightarrow{\delta \rightarrow\!\star\, \mu^R} D \rightarrow\!\!\star\, R$$

- Mnd.-cmnd. int. laws for T,D,R are in bijection with monad maps $T \to D \rightarrow\!\!\star\, R$.

Sweedler theory for duoidal categories

- We follow López Franco and Vasilakopoulou's generalization of Sweedler theory from SMCs to duoidal categories.

- Assume a duoidal category $(\mathbb{D}, I, \diamond, J, \star)$ symm. closed wrt. $(J, \star)$, i.e., with a functor $\multimap_\star \colon \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ such that $- \star G \dashv G \multimap_\star -$.

- The oplax resp. lax monoidal wrt. $(I, \diamond)$ functors

$$\star : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$$
$$\multimap_\star \colon \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$$

  lift to

$$\star : \mathrm{Comon}(\mathbb{D}) \times \mathrm{Comon}(\mathbb{D}) \to \mathrm{Comon}(\mathbb{D}) \quad \text{tensor of comonoids}$$
$$\multimap_\star \colon (\mathrm{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathrm{Mon}(\mathbb{D}) \to \mathrm{Mon}(\mathbb{D}) \quad \textit{power}$$

- A *measuring map* for a monoid T, comonoid D, monoid R (= a mon.-comon. int. law) is a map $UT \star UD \to UR$ whose transpose $T \to D \multimap_\star R$ is a monoid map.

Sweedler theory for duoidal categories ctd.

- If the appropriate adjoints exist, one moreover has functors

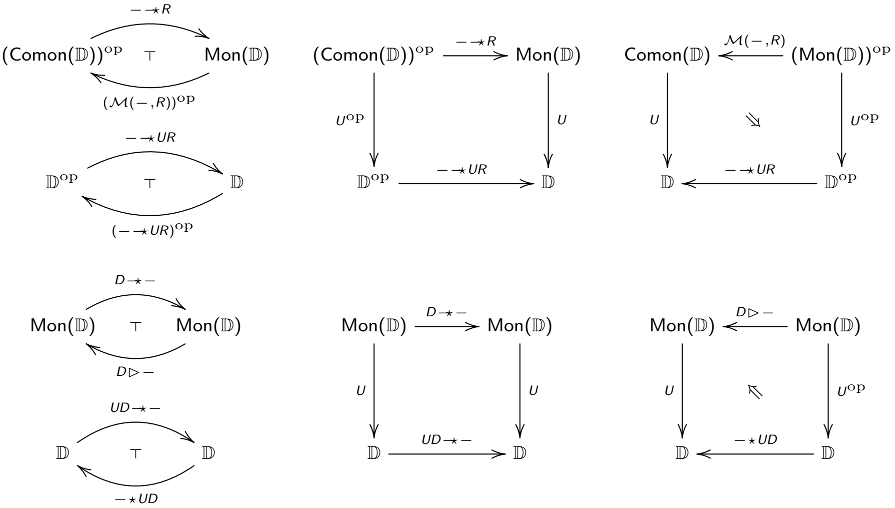
$$
\begin{array}{rll}
\mathcal{C} : (\mathsf{Comon}(\mathbb{D}))^{\mathrm{op}} \times \mathsf{Comon}(\mathbb{D}) \to \mathsf{Comon}(\mathbb{D}) & \text{int. hom of comonoids} \\
\triangleright : \mathsf{Comon}(\mathbb{D}) \times \mathsf{Mon}(\mathbb{D}) \to \mathsf{Mon}(\mathbb{D}) & \textit{Sweedler copower} \\
\mathcal{M} : (\mathsf{Mon}(\mathbb{D}))^{\mathrm{op}} \times \mathsf{Mon}(\mathbb{D}) \to \mathsf{Comon}(\mathbb{D}) & \textit{Sweedler hom}
\end{array}
$$

$$
\frac{D_0 \star D_1 \to D \text{ in } \mathsf{Comon}(\mathbb{D})}{D_0 \to \mathcal{C}(D_1, D) \text{ in } \mathsf{Comon}(\mathbb{D})}
\qquad
\begin{array}{c}
T \to D \mathbin{-\!\star} R \text{ in } \mathsf{Mon}(\mathbb{D}) \\ \hline\hline
UT \star UD \to UR \text{ measuring in } \mathbb{D} \\ \hline\hline
D \triangleright T \to R \text{ in } \mathsf{Mon}(\mathbb{D}) \\ \hline\hline
D \to \mathcal{M}(T, R) \text{ in } \mathsf{Comon}(\mathbb{D})
\end{array}
$$

- $D^\circ = D \mathbin{-\!\star} I$ is called the *dual* of D,
  $D^\bullet = \mathcal{M}(T, I)$ is called the *Sweedler dual* of T.

- The category $(\mathsf{Comon}(\mathbb{D}), J, \star, \mathcal{C})$ is symmetric monoidal closed.
- The category $(\mathsf{Mon}(\mathbb{D}), \mathcal{M}, \triangleright, \mathbin{-\!\star})$ is enriched, copowered and powered over $(\mathsf{Comon}(\mathbb{D}), J, \star, \mathcal{C})$.

Sweedler theory for duoidal categories ctd.



Final interacting (co)monoids, initial residual monoid

- By construction,

  $D \multimap\!\!\star\, R$ is the final monoid T that D interacts R-residually with,
  $\mathcal{M}(T, R)$ is the final comonoid D that T interacts R-residually with,
  $D \triangleright T$ is the initial monoid R wrt. which T and D interact residually.

- $D \multimap\!\!\star\, R$ is immediate to compute since $U(D \multimap\!\!\star\, R) = UD \multimap\!\!\star\, UR$.

- Specifically for $\mathbb{D} = [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ we have

$$(D \multimap\!\!\star\, R)X = \int_Y DY \multimap R(X \otimes Y)$$

  (suppressing the U's).

Mon-comon. int. laws of free monoids

- Exploiting the Sweedler theory perspective, some things about monoid-comonoid interaction become very easy to calculate.

- E.g., for $T = F^*$ (the free monoid on an object F), mon.-comon. int. laws for T, D, R are in bijection with obj.-obj. int. laws for F, UD, UR:

$$
\begin{array}{c}
F \star UD \to UR \text{ in } \mathbb{D} \\
\hline\hline
F \to UD \mathrel{-\!\!\star} UR \text{ in } \mathbb{D} \\
\hline\hline
F \to U(D \mathrel{-\!\!\star} R) \text{ in } \mathbb{D} \\
\hline\hline
F^* \to D \mathrel{-\!\!\star} R \text{ in } \mathrm{Mon}(\mathbb{D}) \\
\hline\hline
U(F^*) \star UD \to UR \text{ measuring in } \mathbb{D}
\end{array}
$$

Sweedler hom from a free monoid

- The Sweedler hom $\mathcal{M}(F^*, R)$ is $(F \multimap\!\star\, UR)^\dagger$ (a cofree comonoid):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D \to (F \multimap\!\star\, UR)^\dagger \text{ in } \mathrm{Comon}(\mathbb{D})}{UD \to F \multimap\!\star\, UR \text{ in } \mathbb{D}}}{F \to UD \multimap\!\star\, UR \text{ in } \mathbb{D}}}{F \to U(D \multimap\!\star\, R) \text{ in } \mathbb{D}}}{F^* \to D \multimap\!\star\, R \text{ in } \mathrm{Mon}(\mathbb{D})}}{D \to \mathcal{M}(F^*, R) \text{ in } \mathrm{Comon}(\mathbb{D})}$$

- For $FX = 1 + X^2$, we have $F^*X \cong \mu X'.\, X + 1 + X'^2$.
  We can calculate $(F \multimap\!\star\, UR)Y \cong R0 + R(2 \times Y)$.
  So $\mathcal{M}(F^*, R)Y \cong \nu Y'.\, Y \times R0 \times R(2 \times Y')$.
  For $RZ = Z$, this means $\mathcal{M}(F^*, R)Y \cong 0$.
  For $RZ = Z + 1$, we get $\mathcal{M}(F^*, R)Y \cong \nu Y'.\, Y \times (2 \times Y' + 1)$.

Sweedler copower of a free monoid

- Similarly, the Sweedler copower $D \triangleright F^*$ is $(F \star UD)^*$ (a free monoid):

$$\dfrac{(F \star UD)^* \to R \text{ in } \mathrm{Mon}(\mathbb{D})}{\dfrac{F \star UD \to UR \text{ in } \mathbb{D}}{\dfrac{F \to UD \multimap_\star UR \text{ in } \mathbb{D}}{\dfrac{F \to U(D \multimap_\star R) \text{ in } \mathbb{D}}{\dfrac{F^* \to D \multimap_\star R \text{ in } \mathrm{Mon}(\mathbb{D})}{D \triangleright F^* \to R \text{ in } \mathrm{Mon}(\mathbb{D})}}}}}$$

- For $FX = 1 + X^2$, we have $F^* X \cong \mu X'.\, X + 1 + X'^2$.
  We can calculate $(F \star UD)X \cong D1 + D(Z^2)$.
  So $(D \triangleright F^*)Z \cong \mu Z'.\, Z + D1 + D(Z'^2)$.

The general case?

- But how to construct $\mathcal{M}(T, R)$ and $D \triangleright T$ nicely and usefully for a general non-free monoid T?
- One possibility is a construction for coequalizers in Mon($\mathbb{D}$).
- We look at a construction for monoids in $\mathbb{D} = [\mathbb{C}, \mathbb{C}]_{\mathrm{a}}$ using a (co)algebraic approach.

A (co)algebraic view

- Mnd.-cmnd. int. laws are in a bijection with coalgebra-algebra internal-homming functors:

$$T(Y \multimap Z) \to DY \multimap RZ \text{ nat. in } Y, Z \text{ subj. to eqs.}$$

$$\begin{array}{ccc} (\mathrm{coEM}(D))^{\mathrm{op}} \times \mathrm{EM}(R) & \longrightarrow & \mathrm{EM}(T) \\ \big\downarrow {\scriptstyle U^{\mathrm{op}} \times U} & & \big\downarrow {\scriptstyle U} \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} & \xrightarrow{\;\multimap\;} & \mathbb{C} \end{array}$$

$$\begin{array}{ccc} (\mathrm{coKl}(D))^{\mathrm{op}} \times \mathrm{Kl}(R) & \longrightarrow & \mathrm{EM}(T) \\ \big\downarrow {\scriptstyle K^{\mathrm{op}} \times K} & & \big\downarrow {\scriptstyle U} \\ \mathbb{C}^{\mathrm{op}} \times \mathbb{C} & \xrightarrow{\;\multimap\;} & \mathbb{C} \end{array}$$

A (co)algebraic view ctd.

- Explicitly, given a mnd.-cmnd. int. law ψ,

  the corresponding (co)alg. exp. functor E sends a coalgebra (Y, χ) of D and an algebra (Z, ζ) of R to the algebra $(Y \multimap Z, \xi)$ of T where

$$\xi = \; T(Y \multimap Z) \xrightarrow{\psi_{Y,Z}} DY \multimap RZ \xrightarrow{\chi \multimap \zeta} Y \multimap Z$$

- Conversely, given a (co)alg. exp. functor E,

  the corresponding mnd.-cmnd. int. law is

$$\psi_{Y,Z} = \; T(Y \multimap Z) \xrightarrow{T(\varepsilon_Y \multimap \eta^R_Z)} T(DY \multimap RZ) \xrightarrow{e_{Y,Z}} DY \multimap RZ$$

  where $(DY \multimap RZ, e_{Y,Z}) = E((DY, \delta_Y), (RZ, \mu^R_Z))$.

Intermediate views

- In fact, the picture is finer, there are also two intermediate bijections:

$$\mathsf{MCIL}_{D,R}(T)$$

$$\cong \qquad\qquad \cong$$

$$[(\mathsf{coEM}(D))^{\mathrm{op}}, (\mathsf{SRun}_R(T))^{\mathrm{op}}]_{\mathrm{cp.}} \qquad [\mathsf{EM}(R), \mathsf{CRun}_D(T)]_{\mathrm{cp.}}$$

$$\cong \qquad\qquad \cong$$

$$[(\mathsf{coEM}(D)^{\mathrm{op}} \times \mathsf{EM}(R), \mathsf{EM}(T)]_{\mathrm{ch.}}$$

where

$\mathsf{MCIL}_{D,R}(T)$ - interaction laws of T, D, R
$\mathsf{SRun}_R(T)$ - R-residual stateful runners of T
$\mathsf{CRun}_D(T)$ - D-fuelled continuation-based runners of T
cp. - preserving carriers
ch. - internal-homming carriers

Stateful runners

- For any Y, we have

$$\frac{\dfrac{\begin{array}{c}\textit{R-residual stateful runners}\text{ of } T \text{ w/ carrier } Y\text{, ie.}\\ TX \times Y \to R(X \times Y) \text{ nat. in } X \text{ subj. to eqs.}\end{array}}{\begin{array}{c}\text{monad morphisms from } T \text{ to } \mathsf{St}_Y^R\text{, ie.}\\ TX \to Y \multimap R(X \times Y) \text{ nat. in } X \text{ subj. to eqs.}\end{array}}}{\begin{array}{ccc}\mathsf{EM}(R) & \longrightarrow & \mathsf{EM}(T)\\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U}\\ \mathbb{C} & \xrightarrow{Y \multimap -} & \mathbb{C}\end{array}}$$

where St_Y^R is the *R-transformed state monad* for state object Y, given by

$$\mathsf{St}_Y^R X = Y \multimap R(X \times Y)$$

Stateful runners ctd.

- More informatively (also characterizing stateful runner maps), $\mathsf{SRun}_R(T)$ is the following pullback in CAT:

$$
\begin{array}{ccccc}
\mathsf{SRun}_R(T) & \longrightarrow & & \longrightarrow & [\mathsf{EM}(R), \mathsf{EM}(T)]^{\mathrm{op}} \\
\Big\downarrow{\scriptstyle U} & & & & \Big\downarrow{\scriptstyle [\mathsf{EM}(R), U]^{\mathrm{op}}} \\
\mathbb{C} & \xrightarrow{(Y \mapsto Y \multimap -)^{\mathrm{op}}} & [\mathbb{C}, \mathbb{C}]^{\mathrm{op}} & \xrightarrow{[U, \mathbb{C}]^{\mathrm{op}}} & [\mathsf{EM}(R), \mathbb{C}]^{\mathrm{op}}
\end{array}
$$

- If U is comonadic, then by the univ. property of $\mathcal{M}(T, R)$ this pullback is also the coEM category of $\mathcal{M}(T, R)$.
- If $\mathbb{C}$ is locally presentable and T, R are accessible, which we assume, then U is comonadic.

- Eg., for $TX = S \Rightarrow X$ (the reader monad), $RZ = Z$, we have $\mathsf{SRun}_R(T) \cong \mathbb{C}/S \cong \mathsf{coEM}(D)$ where $DY = S \times Y$ (the coreader comonad).

  The same holds for $RZ = Z + 1$.

Continuation-based runners ctd.

- For any Z, we have

D-fuelled continuation-based runners of T w/ carrier Z, ie.

$D(X \multimap Z) \to TX \multimap Z$ nat. in X subj. to eqs.

$$\overline{\overline{}}$$

monad morphisms from T to Cnt^D_Z, ie.

$TX \to D(X \multimap Z) \multimap Z$ nat. in X subj. to eqs.

$$\overline{\overline{}}$$

$$\begin{array}{ccc} (\mathsf{coEM}(D))^{\mathrm{op}} & \longrightarrow & \mathsf{EM}(T) \\ {\scriptstyle U^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbb{C}^{\mathrm{op}} & \xrightarrow{\;-\multimap Z\;} & \mathbb{C} \end{array}$$

where Cnt^D_Z is the *D-transformed continuation monad* for answer object Z, given by

$$\mathsf{Cnt}^D_Z X = D(X \multimap Z) \multimap Z$$

Continuation-based runners ctd.

- Moreover, $\mathsf{CRun}_D(T)$ is this pullback:

$$
\begin{array}{ccccc}
\mathsf{CRun}_D(T) & \longrightarrow & & & [(\mathsf{coEM}(D))^{\mathrm{op}}, \mathsf{EM}(T)] \\
{\scriptstyle U}\downarrow & & & & \downarrow{\scriptstyle [(\mathsf{coEM}(D))^{\mathrm{op}}, U]} \\
\mathbb{C} & \xrightarrow{Z \mapsto - \multimap Z} & [\mathbb{C}^{\mathrm{op}}, \mathbb{C}] & \xrightarrow{[U^{\mathrm{op}}, \mathbb{C}]} & [(\mathsf{coEM}(D))^{\mathrm{op}}, \mathbb{C}]
\end{array}
$$

- If U is monadic, then by the univ. property of $D \triangleright T$ the same pullback is also the EM category of $D \triangleright T$.
- If $\mathbb{C}$ is locally presentable and T, D are accessible, which we assume, then U is monadic.

Not today

- Strong (enriched) monad-comonad int. laws

- For $\mathbb{V}$ a monoidal category acting on $\mathbb{C}$, T a $\mathbb{V}$-strong monad on $\mathbb{V}$, D a $\mathbb{V}$-strong comonad on $\mathbb{C}$, R a $\mathbb{V}$-strong monad on $\mathbb{C}$,
  an int. law. is a $\mathbb{V}$-strong nat. transf. $TX \bullet DY \to R(X \bullet Y)$.

- Int. laws for (co)monads given by (co)models of theories

- The Sweedler dual of T induced by models of a theory
  is induced by comodels of the same theory.

Takeaway

- Functor-functor and monad-comonad interaction laws generalize to object-object and monoid-comonoid interaction laws in duoidal categories.
- Final interacting (co)monoids, initial residual monoids have been studied in algebra, in Sweedler theory.
- The Sweedler theory perspective allows working with interaction laws at a very abstract level.
- For certain calculations specifically for monad-comonad interaction laws, combinationwith the (co)algebraic perspective is helpful.

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