The free bifibration over a functor

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[1] Joint work with Bryce Clarke and Gabriel Scherer. Some parts written up, some in progress.

What is a bifibration?

One category living over another category, such that objects of the category above may be *pushed* and *pulled* along arrows of the category below.

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Formally:

$$\begin{tikzcd} \mathcal{D} \arrow[d, "p"'] \\ \mathcal{C} \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{D} \arrow[d, "p"'] & S \arrow[d, dotted, no head] & \\ \mathcal{C} & A \arrow[r, "f"] & B \end{tikzcd}$$

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\end{array}
$$

Bifibrations as indexed categories and adjunctions

The operations of pushing and pulling along $f : A \to B$ of $\mathcal{C}$ induce an *adjunction*

$$\begin{tikzcd} \mathcal{D}_A \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathcal{D}_B \arrow[l, bend left, "f^*"] \end{tikzcd}$$

between the *fiber categories* $\mathcal{D}_A = p^{-1}(\mathrm{id}_A)$ and $\mathcal{D}_B = p^{-1}(\mathrm{id}_B)$.

This observation extends to an equivalence between bifibrations $\mathcal{D} \to \mathcal{C}$ and *pseudofunctors* $\mathcal{C} \to \mathcal{A}\mathrm{dj}$ *into the category of small categories and adjunctions.*

A few examples (from logic and computer science)

1. The forgetful functor $\mathcal{S}\mathrm{ubset} \to \mathcal{S}\mathrm{et}$ is a bifibration, where:

$$f_*(S \subseteq A) = f(S) \qquad f^*(T \subseteq B) = f^{-1}(T)$$

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2. The functor $p : \mathcal{R}\mathrm{el}_\bullet \to \mathcal{R}\mathrm{el}$ is a bifibration, where:

$$r_*(S \subseteq A) = \{\, b \mid \exists a.\, (a,b) \in r \wedge a \in S \,\} \quad (= \text{“}\Diamond_r S\text{”})$$
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3. A functor $p : \mathcal{Q} \to \mathcal{B}\Sigma$ representing a NDFA is a bifibration just in case the automaton is both (total) deterministic and codeterministic.

$$\begin{tikzcd} & q_1 \arrow[dr, "a"] & \\ q_0 \arrow[ur, "a"] & & q_2 \arrow[ll, "a"] \end{tikzcd}$$

Problem

Given a functor, can we turn it into a bifibration in a universal way?

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Okay! But how to construct Λ_p? This question has been relatively little-studied:

- R. Dawson, R. Paré, and D. Pronk. Adjoining adjoints. *Adv. Mathematics*, 178(1):99–140, 2003.
- François Lamarche. Path functors in Cat. Unpublished, 2010. HAL-00831430.

Our work

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Finally, we found a couple *nice examples* of free bifibrations of a combinatorial nature.

A sequent calculus for $\mathcal{B}\mathrm{if}(p)$

Formulas and derivations

Bifibrational formulas: $(S \sqsubset A)$

$$\frac{X \in \mathcal{D} \quad p(X) = A}{X \sqsubset A} \qquad \frac{S \sqsubset A \quad f : A \to B}{f_* S \sqsubset B} \qquad \frac{f : A \to B \quad T \sqsubset B}{f^* T \sqsubset A}$$

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$$\frac{\delta : X \to Y \in \mathcal{D} \qquad p(\delta) = f}{X \underset{f}{\Longrightarrow} Y}\,\delta$$

Equational theory on derivations

Consider four **permutation equivalences** on derivations, including

$$\dfrac{\dfrac{S \underset{fg}{\Longrightarrow} T}{S \underset{fgh}{\Longrightarrow} h_* \, T}\ \mathsf{R}h_*}{f_* \, S \underset{gh}{\Longrightarrow} h_* \, T}\ \mathsf{L}f_* \sim \dfrac{\dfrac{S \underset{fg}{\Longrightarrow} T}{f_* \, S \underset{g}{\Longrightarrow} T}\ \mathsf{L}f_*}{f_* \, S \underset{gh}{\Longrightarrow} h_* \, T}\ \mathsf{R}h_* \qquad \dfrac{\dfrac{S \underset{g}{\Longrightarrow} T}{S \underset{gh}{\Longrightarrow} h_* \, T}\ \mathsf{R}h_*}{f^* \, S \underset{fgh}{\Longrightarrow} h_* \, T}\ \mathsf{L}f^* \sim \dfrac{\dfrac{S \underset{g}{\Longrightarrow} T}{S^* \underset{fg}{\Longrightarrow} T}\ \mathsf{L}f^*}{f^* \, S \underset{fgh}{\Longrightarrow} h_* \, T}\ \mathsf{R}h_*$$

plus their symmetric versions with pushforward and pullback swapped.

Example derivations

$$\begin{array}{ccc} \mathcal{D} & & X \xrightarrow{\alpha} Y \\ \big\downarrow p & & \\ \mathcal{C} & & A \xrightarrow{f} B \xrightarrow{g} C \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \underset{f}{\Longrightarrow} Y}\,\alpha}{X \underset{f\;g}{\Longrightarrow} g_* \, Y}\,\mathsf{R}g^*}{f_* \, X \underset{g}{\Longrightarrow} g_* \, Y}\,\mathsf{L}f_*}{f_* \, X \underset{\mathsf{id}_B}{\Longrightarrow} g^* \, g_* \, Y}\,\mathsf{R}g^* \quad \sim \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X \underset{f}{\Longrightarrow} Y}\,\alpha}{X \underset{f\;g}{\Longrightarrow} g_* \, Y}\,\mathsf{R}g^*}{X \underset{g}{\Longrightarrow} g^* \, g_* \, Y}\,\mathsf{R}g^*}{f_* \, X \underset{\mathsf{id}_B}{\Longrightarrow} g^* \, g_* \, Y}\,\mathsf{L}f_*$$

Putting it all together

Let $\mathcal{B}\mathrm{if}(p)$ be the category whose objects are bifibrational formulas and whose arrows are $\sim$-equivalence classes of derivations, with composition defined by cut-elimination. (Non-trivial to show this is a category!)

Let Λ_p be the functor $\mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ sending $S \sqsubset A$ to A and $\alpha : S \underset{f}{\Longrightarrow} T$ to f.

Theorem. $\Lambda_p : \mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ is the free bifibration on $p : \mathcal{D} \to \mathcal{C}$.

The double category of zigzags

Definition

Given a category $\mathcal{C}$, the double category of zigzags $\mathbb{Z}\mathcal{C}$ is defined as follows:

- objects are the objects of $\mathcal{C}$
- horizontal arrows are the arrows of $\mathcal{C}$
- vertical arrows are (not necessarily strictly alternating) zigzags of arrows in $\mathcal{C}$
- double cells are generated by vertical pastings of four families of cells

$$
\begin{tikzcd}
\cdot \arrow[r, "fg"] \arrow[d, "f"'] \arrow[dr, phantom, "L"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[dr, phantom, "\overline{L}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[u, "f"] \arrow[r, "fg"'] & \cdot
\end{tikzcd}
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\cdot \arrow[r, "fg"'] & \cdot
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\cdot \arrow[r, "fg"] \arrow[d, equal] \arrow[dr, phantom, "\overline{R}"] & \cdot \\
\cdot \arrow[r, "f"'] & \cdot \arrow[u, "g"']
\end{tikzcd}
$$

modulo four relations...

Definition

- …relations including

$$
\begin{tikzcd}
\cdot \arrow[r, "fg"] \arrow[d, "f"'] \arrow[dr, phantom, "L"] & \cdot \arrow[d, equal] \\
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\end{tikzcd}
\sim
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\cdot \arrow[r, "fg"] \arrow[d, equal] \arrow[dr, phantom, "R"] & \cdot \arrow[d, "h"] \\
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\end{tikzcd}
$$

  plus their symmetric versions with the vertical arrows flipped
- vertical composition is just concatenation of stacks of generators
- horizontal composition is defined in a more complicated way, by gluing vertical stacks of generators along their boundaries and reducing appropriately

Relation to $\mathcal{B}\mathrm{if}(p)$

Like any double category, $\mathbb{Z}\mathcal{C}$ may be viewed as an internal category in $\mathcal{C}\mathrm{at}$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "\epsilon" description] & \mathcal{Z}\mathcal{C} \arrow[l, shift right=4ex, "\mathsf{src}"'] \arrow[l, shift left=4ex, "\mathsf{tgt}"] & \mathcal{Z}\mathcal{C} \times_{\mathcal{C}} \mathcal{Z}\mathcal{C} \arrow[l, "\odot"']
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where $\mathcal{Z}\mathcal{C}$ is the category whose objects are the vertical arrows ($=$ zigzags), and whose arrows are double cells, composed horizontally.

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Fact: $\mathsf{tgt} : \mathcal{Z}\mathcal{C} \to \mathcal{C}$ is the free bifibration over $\mathsf{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$! (And so is src.)

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Conversely, any free bifibration can be constructed as follows:

$$\begin{tikzcd} \mathcal{D} \arrow[d, "p"'] & & \\ \mathcal{C} & \mathcal{Z}\mathcal{C} \arrow[l, "\mathsf{src}"] \arrow[r, "\mathsf{tgt}"'] & \mathcal{C} \end{tikzcd}$$

Relation to $\mathcal{B}\mathrm{if}(p)$

Like any double category, $\mathbb{Z}\mathcal{C}$ may be viewed as an internal category in $\mathcal{C}\mathrm{at}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\epsilon" description] & \mathcal{Z}\mathcal{C} \arrow[l, bend right, "\mathsf{src}"'] \arrow[l, bend left, "\mathsf{tgt}"] & \mathcal{Z}\mathcal{C} \times_{\mathcal{C}} \mathcal{Z}\mathcal{C} \arrow[l, "\odot"']
\end{tikzcd}$$

where $\mathcal{Z}\mathcal{C}$ is the category whose objects are the vertical arrows ($=$ zigzags), and whose arrows are double cells, composed horizontally.

Fact: $\mathsf{tgt} : \mathcal{Z}\mathcal{C} \to \mathcal{C}$ is the free bifibration over $\mathsf{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$! (And so is src.)

Conversely, any free bifibration can be constructed as follows:

$$\begin{tikzcd}
\mathcal{D} \arrow[d, "p"'] & \mathcal{B}\mathrm{if}(p) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \\
\mathcal{C} & \mathcal{Z}\mathcal{C} \arrow[l, "\mathsf{src}"] \arrow[r, "\mathsf{tgt}"'] & \mathcal{C}
\end{tikzcd}$$

Relation to $\mathcal{B}\mathrm{if}(p)$

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\epsilon" description] & \mathcal{Z}\mathcal{C} \arrow[l, shift right=3, "\mathsf{src}"'] \arrow[l, shift left=3, "\mathsf{tgt}"] & \mathcal{Z}\mathcal{C} \times_{\mathcal{C}} \mathcal{Z}\mathcal{C} \arrow[l, "\odot"']
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$$\begin{tikzcd}
\mathcal{D} \arrow[d, "p"'] \arrow[dr, phantom, "\llcorner"] & \mathcal{B}\mathrm{if}(p) \arrow[l] \arrow[d] \arrow[dr, "\Lambda_p"] & \\
\mathcal{C} & \mathcal{Z}\mathcal{C} \arrow[l, "\mathsf{src}"] \arrow[r, "\mathsf{tgt}"'] & \mathcal{C}
\end{tikzcd}$$

Adjoining adjoints

Dawson, Paré, and Pronk posed (and solved) the problem of constructing a 2-category $\Pi_2\mathcal{C}$ by freely adjoining right adjoints to all the arrows of a category $\mathcal{C}$.

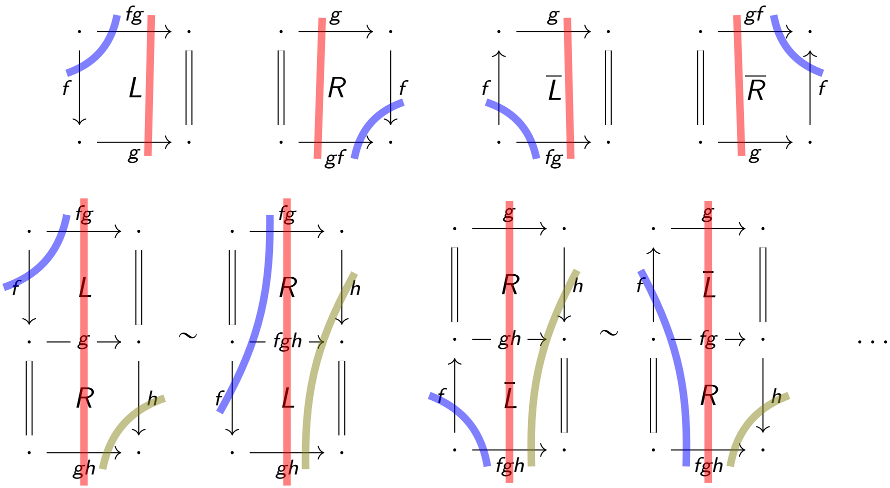
The double category of zigzags provides another solution to this problem.

Indeed, $\Pi_2\mathcal{C}$ can be obtained as the underlying *vertical 2-category* of $\mathbb{Z}\mathcal{C}$, consisting of the objects, vertical arrows, and double cells framed by horizontal identities.

(Thus all three objects $\mathcal{B}\mathrm{if}(p)$, $\mathbb{Z}\mathcal{C}$, $\Pi_2\mathcal{C}$ are closely related!)

For the case when $\mathcal{C}$ is a free category, DPP also introduced a graphical representation of 2-cells in $\Pi_2\mathcal{C}$ as certain planar diagrams. These diagrams may be neatly recovered as *string diagrams* for the double category $\mathbb{Z}\mathcal{C}$ (cf. Jaz Myers 2018).

String diagrams



Now for some examples!

Example #1

Consider the following functor:

$$\begin{tikzcd} 1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] & \\ 2 & 0 \arrow[r, "f"] & 1 \end{tikzcd}$$

Puzzle: what is the free bifibration over p? Hmm...

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Objects in $\mathcal{B}\mathrm{if}(p)_0$ are isomorphic to even-length sequences $S \equiv f^* f_* \cdots f^* f_* 0$

Example #1

Consider the following functor:

$$\begin{array}{ccccc} 1 & & 0 & & \\ {\scriptstyle p}\downarrow & & \vdots & & \\ 2 & & 0 & \xrightarrow{f} & 1 \end{array}$$

Puzzle: what is the free bifibration over p? Hmm...

Objects in $\mathcal{B}\mathrm{if}(p)_0$ are isomorphic to even-length sequences $S \equiv f^* \, f_* \cdots f^* \, f_* \, 0$

Objects in $\mathcal{B}\mathrm{if}(p)_1$ are isomorphic to odd-length sequences $T \equiv f_* \cdots f^* \, f_* \, 0$

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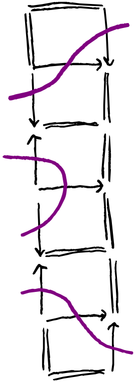
Objects in $\mathcal{B}\mathrm{if}(p)_0$ are isomorphic to even-length sequences $S \equiv f^* \, f_* \cdots f^* \, f_* \, 0$

Objects in $\mathcal{B}\mathrm{if}(p)_1$ are isomorphic to odd-length sequences $T \equiv f_* \cdots f^* \, f_* \, 0$

What are the morphisms in the fibers? It may help to enumerate them...

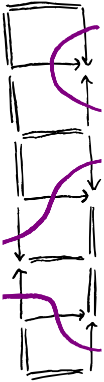
One morphism $2 \to 1$

$$
\begin{array}{cl}
\hline
0 \underset{\mathrm{id}_0}{\Longrightarrow} 0 & id \\
\hline
0 \underset{f}{\Longrightarrow} f_* \, 0 & \mathsf{R}f_* \\
\hline
f_* \, 0 \underset{\mathrm{id}_1}{\Longrightarrow} f_* \, 0 & \mathsf{L}f_* \\
\hline
f^* \, f_* \, 0 \underset{f}{\Longrightarrow} f_* \, 0 & \mathsf{L}f^* \\
\hline
f_* \, f^* \, f_* \, 0 \underset{\mathrm{id}_1}{\Longrightarrow} f_* \, 0 & \mathsf{L}f_* \\
\hline
f^* \, f_* \, f^* \, f_* \, 0 \underset{f}{\Longrightarrow} f_* \, 0 & \mathsf{L}f^* \\
\hline
f^* \, f_* \, f^* \, f_* \, 0 \underset{\mathrm{id}_0}{\Longrightarrow} f^* \, f_* \, 0 & \mathsf{R}f^*
\end{array}
$$

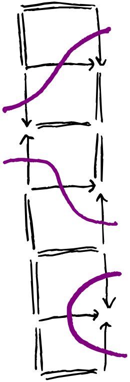


Two morphisms $1 \to 2$

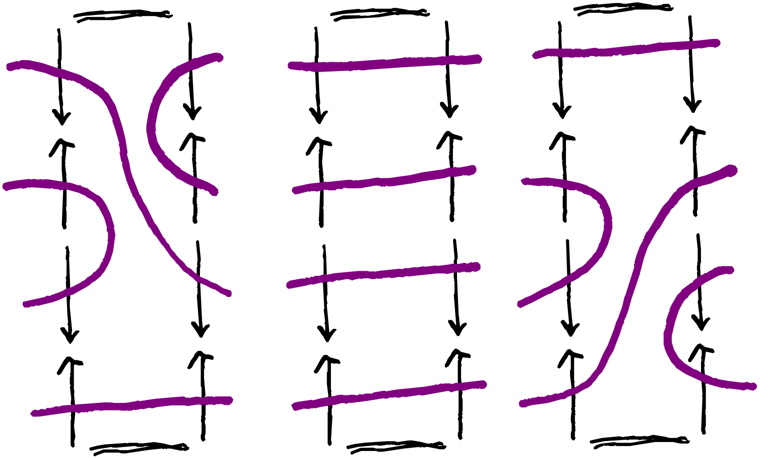
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \underset{\mathrm{id}_0}{\Longrightarrow} 0}\ \mathrm{id}_0}{0 \underset{f}{\Longrightarrow} f_*\, 0}\ \mathrm{R}f_*}{0 \underset{\mathrm{id}_0}{\Longrightarrow} f^*\, f_*\, 0}\ \mathrm{R}f^*}{0 \underset{f}{\Longrightarrow} f_*\, f^*\, f_*\, 0}\ \mathrm{R}f_*}{f_*\, 0 \underset{\mathrm{id}_1}{\Longrightarrow} f_*\, f^*\, f_*\, 0}\ \mathrm{L}f_*}{f^*\, f_*\, 0 \underset{f}{\Longrightarrow} f_*\, f^*\, f_*\, 0}\ \mathrm{L}f^*}{f^*\, f_*\, 0 \underset{\mathrm{id}_0}{\Longrightarrow} f^*\, f_*\, f^*\, f_*\, 0}\ \mathrm{R}f^*
$$



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \underset{\mathrm{id}_0}{\Longrightarrow} 0}\ \mathrm{id}_0}{0 \underset{f}{\Longrightarrow} f_*\, 0}\ \mathrm{R}f_*}{f_*\, 0 \underset{\mathrm{id}_1}{\Longrightarrow} f_*\, 0}\ \mathrm{L}f_*}{f^*\, f_*\, 0 \underset{f}{\Longrightarrow} f_*\, 0}\ \mathrm{L}f^*}{f^*\, f_*\, 0 \underset{\mathrm{id}_0}{\Longrightarrow} f^*\, f_*\, 0}\ \mathrm{R}f^*}{f^*\, f_*\, 0 \underset{f}{\Longrightarrow} f_*\, f^*\, f_*\, 0}\ \mathrm{R}f_*}{f^*\, f_*\, 0 \underset{\mathrm{id}_0}{\Longrightarrow} f^*\, f_*\, f^*\, f_*\, 0}\ \mathrm{R}f^*
$$



Three morphisms $2 \to 2$



$$f^* \, f_* \, f^* \, f_* \, 0 \xRightarrow[\mathrm{id}_0]{} f^* \, f_* \, f^* \, f_* \, 0$$

Punchline #1

Arrows $(f^* f_*)^m 0 \longrightarrow (f^* f_*)^n 0$ in $\mathcal{B}\mathrm{if}(p)_0$ correspond to monotone maps $\mathsf{m} \to \mathsf{n}$!

Indeed, the push-pull adjunction captures the adjunction

$$\begin{tikzcd} \Delta \arrow[rr, bend left, "f_*"] \arrow[rr, phantom, "\perp"] & & \Delta_\perp \arrow[ll, bend left, "f^*"] \end{tikzcd}$$

between the category Δ of finite ordinals and order-preserving maps, and the category $\Delta_\perp$ of non-empty finite ordinals and order-and-least-element-preserving maps.

We were even more surprised that the total category is equivalent $\mathcal{B}\mathrm{if}(p) \cong \Upsilon$ to the *category of schedules* introduced by Harmer, Hyland, and Melliès in their study of the categorical combinatorics of innocent strategies (LICS 2007).

Example #2

Now consider the following functor:

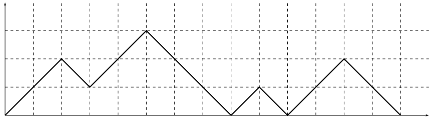
$$
\begin{tikzcd}
1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] \\
\mathbb{N} & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \dots
\end{tikzcd}
$$

Build the free bifibration $\mathcal{B}\mathrm{if}(p) \to \mathbb{N}$, and look at the fiber of 0.

Puzzle: what are its objects?

A category with Dyck walks as objects!

$$f^* \, f^* \, f_* \, f_* \, f^* \, f_* \, f^* \, f^* \, f^* \, f_* \, f_* \, f^* \, f_* \, f_* \, 0 \quad =$$

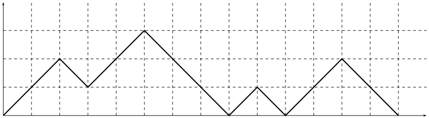


But what is a *morphism* of Dyck walks??

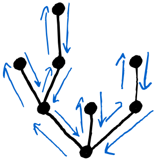
The $\mathcal{B}\mathrm{if}(-)$ construction gives an answer. Is it something natural/known?

Reconstructing the Batanin-Joyal category of trees

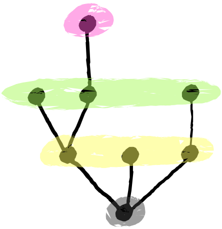
Dyck paths have a well-known, canonical bijection with (finite rooted plane) trees.



$\mapsto$



Trees may also be encoded as *functors* $T : \mathbb{N}^{\mathrm{op}} \to \Delta$.



$$\begin{array}{c} 1 \\ \downarrow \\ 3 \\ \downarrow \\ 3 \\ \downarrow \\ 1 \end{array}$$

Reconstructing the Batanin-Joyal category of trees

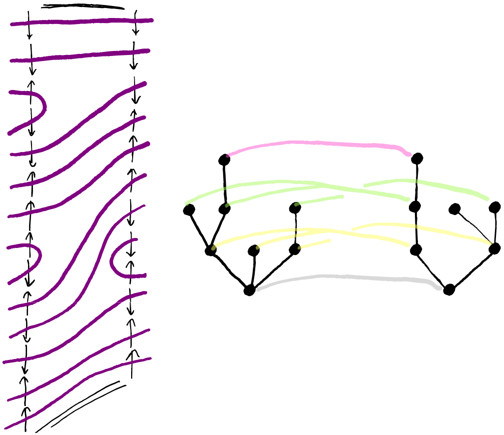
Consider *natural transformations* $\theta : S \Rightarrow T$.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
S(2) \arrow[r, "\theta_2"] \arrow[d] & T(2) \arrow[d] \\
S(1) \arrow[r, "\theta_1"] \arrow[d] & T(1) \arrow[d] \\
S(0) = 1 \arrow[r, equal] & 1 = T(0)
\end{tikzcd}
$$

In other words, map nodes to nodes of the same height, respecting parents.

Punchline #2

Theorem: $\mathcal{B}\mathrm{if}(p : 1 \to \mathbb{N})_0 \cong \mathrm{PTree}$.



(More generally, $\mathcal{B}\mathrm{if}(p)_k \cong \mathrm{PTree}_k =$ category of finite rooted plane trees whose rightmost branch is pointed by a node of height k.)

Summary

We have a clean and simple proof-theoretic construction of free bifibrations, with complentary algebraic & topological perspectives.

Work in progress on characterizing normal forms.

Some surprisingly rich combinatorics emerges as if out of thin air.

$$\begin{tikzcd} 1 \arrow[d, "p"'] & 0 \arrow[d, dotted, no head] \\ \mathbb{N} & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \dots \end{tikzcd}$$