Predicativism, Universality and Low-Complexity Computation

Amir Akbar Tabatabai

Institute of Mathematics, Czech Academy of Sciences

Structure Meets Power 2024, Tallin

The Overview

Recalling F-algebras, we use

The Overview

Recalling F-algebras, we use

- **Philosophy:** absolute predicativism to weaken the notion of

The Overview

Recalling F-algebras, we use

- **Philosophy:** absolute predicativism to weaken the notion of
- **Structure:** an initial F-algebra to a relative notion called a predicative F-scheme and then

The Overview

Recalling F-algebras, we use

- **Philosophy:** absolute predicativism to weaken the notion of
- **Structure:** an initial F-algebra to a relative notion called a predicative F-scheme and then
- **Power:** show how it captures the low-complexity computation.

The Overview

Recalling F-algebras, we use

- **Philosophy:** absolute predicativism to weaken the notion of
- **Structure:** an initial F-algebra to a relative notion called a predicative F-scheme and then
- **Power:** show how it captures the low-complexity computation.

We provide a synthetic/categorical foundation for low-complexity computation.

F-Algebras

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor:

F-Algebras

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor:

- An F-algebra in $\mathcal{C}$ is a pair $\mathfrak{A} = (|\mathfrak{A}|, a_{\mathfrak{A}})$ of an object $|\mathfrak{A}|$ in $\mathcal{C}$ and a map $a_{\mathfrak{A}} : F(|\mathfrak{A}|) \to |\mathfrak{A}|$.

F-Algebras

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor:

- An *F-algebra* in $\mathcal{C}$ is a pair $\mathfrak{A} = (|\mathfrak{A}|, a_{\mathfrak{A}})$ of an object $|\mathfrak{A}|$ in $\mathcal{C}$ and a map $a_{\mathfrak{A}} : F(|\mathfrak{A}|) \to |\mathfrak{A}|$.
- An *F-morphism* $f : \mathfrak{A} \to \mathfrak{B}$ is a $\mathcal{C}$-map $f : |\mathfrak{A}| \to |\mathfrak{B}|$ such that:

$$\begin{tikzcd} F(|\mathfrak{A}|) \arrow[r, "F(f)"] \arrow[d, "a_{\mathfrak{A}}"'] & F(|\mathfrak{B}|) \arrow[d, "a_{\mathfrak{B}}"] \\ |\mathfrak{A}| \arrow[r, "f"'] & |\mathfrak{B}| \end{tikzcd}$$

F-Algebras

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{C}$ be a functor:

- An F-algebra in $\mathcal{C}$ is a pair $\mathfrak{A} = (|\mathfrak{A}|, a_{\mathfrak{A}})$ of an object $|\mathfrak{A}|$ in $\mathcal{C}$ and a map $a_{\mathfrak{A}} : F(|\mathfrak{A}|) \to |\mathfrak{A}|$.
- An F-morphism $f : \mathfrak{A} \to \mathfrak{B}$ is a $\mathcal{C}$-map $f : |\mathfrak{A}| \to |\mathfrak{B}|$ such that:

$$\begin{tikzcd} F(|\mathfrak{A}|) \arrow[r, "F(f)"] \arrow[d, "a_{\mathfrak{A}}"'] & F(|\mathfrak{B}|) \arrow[d, "a_{\mathfrak{B}}"] \\ |\mathfrak{A}| \arrow[r, "f"'] & |\mathfrak{B}| \end{tikzcd}$$

Denote the category of F-algebras in $\mathcal{C}$ by $\mathrm{Alg}_F(\mathcal{C})$. We are mainly interested in the two functors $F_{\mathbb{N}}(X) = 1 + X$ and $F_{\mathbb{W}}(X) = 1 + X + X$.

Blueprints

What is the blueprint of all F-algebras in $\mathcal{C}$, specifying all the things an F-algebra in $\mathcal{C}$ must satisfy and nothing more?

Blueprints

What is the blueprint of all F-algebras in $\mathcal{C}$, specifying all the things an F-algebra in $\mathcal{C}$ must satisfy and nothing more? Initial F-algebra, if exists.

Blueprints

What is the blueprint of all F-algebras in $\mathcal{C}$, specifying all the things an F-algebra in $\mathcal{C}$ must satisfy and nothing more? Initial F-algebra, if exists.

- For $\mathcal{C} = \text{Set}$, the initial $F_{\mathbb{N}}$-algebra is the structure $(\mathbb{N}, 0, s)$ of natural numbers.

Blueprints

What is the blueprint of all F-algebras in $\mathcal{C}$, specifying all the things an F-algebra in $\mathcal{C}$ must satisfy and nothing more? Initial F-algebra, if exists.

- For $\mathcal{C} = \text{Set}$, the initial $F_{\mathbb{N}}$-algebra is the structure $(\mathbb{N}, 0, s)$ of natural numbers.
- For $\mathcal{C} = \text{Set}$, the initial $F_{\mathbb{W}}$-algebra is the structure $(\mathbb{W}, \epsilon, s^0, s^1)$ of binary strings, where ϵ is the empty string, $s^0(w) = w0$ and $s^1(w) = w1$.

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.
- We need to assume the existence of a reasonable inductive set (A, a, f) and define $\mathbb{N}$ as the intersection of all inductive subsets of (A, a, f).

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.
- We need to assume the existence of a reasonable inductive set (A, a, f) and define $\mathbb{N}$ as the intersection of all inductive subsets of (A, a, f).
- The range of the intersection is powerset-like and hence huge.

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.
- We need to assume the existence of a reasonable inductive set (A, a, f) and define $\mathbb{N}$ as the intersection of all inductive subsets of (A, a, f).
- The range of the intersection is powerset-like and hence huge.
- If we don't accept the set of all subsets in the first place, this powerset-like range includes $\mathbb{N}$ and hence the definition somehow refers to itself which is a vicious circle.

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.
- We need to assume the existence of a reasonable inductive set (A, a, f) and define $\mathbb{N}$ as the intersection of all inductive subsets of (A, a, f).
- The range of the intersection is powerset-like and hence huge.
- If we don't accept the set of all subsets in the first place, this powerset-like range includes $\mathbb{N}$ and hence the definition somehow refers to itself which is a vicious circle.

How to solve the issue?

Impredicativity

The initial F-algebra is a reasonable choice for the blueprint of all F-algebras. However:

- It is defined impredicatively! Why?
- Think of the set of natural numbers. We cannot define it from below.
- We need to assume the existence of a reasonable inductive set (A, a, f) and define $\mathbb{N}$ as the intersection of all inductive subsets of (A, a, f).
- The range of the intersection is powerset-like and hence huge.
- If we don't accept the set of all subsets in the first place, this powerset-like range includes $\mathbb{N}$ and hence the definition somehow refers to itself which is a vicious circle.

How to solve the issue? Stratification!

Let $\mathcal{D}$ be a category, $\mathcal{C}$ be its subcategory, $i : \mathcal{C} \to \mathcal{D}$ be the inclusion functor and $F : \mathcal{D} \to \mathcal{D}$ be a functor whose restriction to $\mathcal{C}$ lands in $\mathcal{C}$.

Let $\mathcal{D}$ be a category, $\mathcal{C}$ be its subcategory, $i : \mathcal{C} \to \mathcal{D}$ be the inclusion functor and $F : \mathcal{D} \to \mathcal{D}$ be a functor whose restriction to $\mathcal{C}$ lands in $\mathcal{C}$.

The F-scheme I of $\mathcal{C}$ in $\mathcal{D}$ is the limit in $\mathcal{D}$ of the diagram consisting of the underlying objects of F-algebras in $\mathcal{C}$ together with F-morphisms between them in $\mathcal{C}$:

$$\begin{tikzcd}
& I \arrow[dl, "r_{\mathfrak{A}}"'] \arrow[dr, "r_{\mathfrak{B}}"] \arrow[dd, "r_{\mathfrak{C}}"] & \\
|\mathfrak{A}| \arrow[rr, "f"'] \arrow[dr, "h"'] & & |\mathfrak{B}| \arrow[dl, "g"] \\
& |\mathfrak{C}| &
\end{tikzcd}$$

Let $\mathcal{D}$ be a category, $\mathcal{C}$ be its subcategory, $i : \mathcal{C} \to \mathcal{D}$ be the inclusion functor and $F : \mathcal{D} \to \mathcal{D}$ be a functor whose restriction to $\mathcal{C}$ lands in $\mathcal{C}$.

The F-scheme I of $\mathcal{C}$ in $\mathcal{D}$ is the limit in $\mathcal{D}$ of the diagram consisting of the underlying objects of F-algebras in $\mathcal{C}$ together with F-morphisms between them in $\mathcal{C}$:

$$\begin{tikzcd}
& I \arrow[dl, "r_{\mathfrak{A}}"'] \arrow[dd, "r_{\mathfrak{C}}"] \arrow[dr, "r_{\mathfrak{B}}"] & \\
|\mathfrak{A}| \arrow[rr, "f"'] \arrow[dr, "h"'] & & |\mathfrak{B}| \arrow[dl, "g"] \\
& |\mathfrak{C}| &
\end{tikzcd}$$

Formally, I is the limit of the diagram $i|-| : \mathrm{Alg}_F(\mathcal{C}) \to \mathcal{D}$ in $\mathcal{D}$.

Some Remarks

- A construction is algebraic if it is preserved under all algebraic morphisms.

Some Remarks

- A construction is algebraic if it is preserved under all algebraic morphisms.
- The limit extracts out all algebraic constructions in $\mathcal{C}$ in one object. Hence, the limit can be called the bluprint or the scheme from which the algebraic constructions in $\mathcal{C}$ are constructed.

Some Remarks

- A construction is algebraic if it is preserved under all algebraic morphisms.
- The limit extracts out all algebraic constructions in $\mathcal{C}$ in one object. Hence, the limit can be called the bluprint or the scheme from which the algebraic constructions in $\mathcal{C}$ are constructed.
- For $\mathcal{C} = \mathrm{Set}$ and $\mathcal{D} = \mathrm{Class}$, the F-scheme I is the class of the sequences like $\langle e_{\mathfrak{A}} \rangle_{\mathfrak{A}}$ such that $e_{\mathfrak{A}} \in |\mathfrak{A}|$ and $f(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$, for any F-morphism $f : \mathfrak{A} \to \mathfrak{B}$. The bluprint $\langle e_{\mathfrak{A}} \rangle_{\mathfrak{A}}$ dictates the way we must construct an algebraic construction in $\mathfrak{A}$, for any F-algebra $\mathfrak{A}$ in $\mathcal{C}$.

Some Remarks

- A construction is algebraic if it is preserved under all algebraic morphisms.
- The limit extracts out all algebraic constructions in $\mathcal{C}$ in one object. Hence, the limit can be called the bluprint or the scheme from which the algebraic constructions in $\mathcal{C}$ are constructed.
- For $\mathcal{C} = \mathrm{Set}$ and $\mathcal{D} = \mathrm{Class}$, the F-scheme I is the class of the sequences like $\langle e_{\mathfrak{A}} \rangle_{\mathfrak{A}}$ such that $e_{\mathfrak{A}} \in |\mathfrak{A}|$ and $f(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$, for any F-morphism $f : \mathfrak{A} \to \mathfrak{B}$. The bluprint $\langle e_{\mathfrak{A}} \rangle_{\mathfrak{A}}$ dictates the way we must construct an algebraic construction in $\mathfrak{A}$, for any F-algebra $\mathfrak{A}$ in $\mathcal{C}$.
- For $F = F_{\mathbb{N}}$, the element $2 = \langle s_{\mathfrak{A}} s_{\mathfrak{A}} 0_{\mathfrak{A}} \rangle_{\mathfrak{A}}$ is the blueprint for the two iterations of a generic function on a generic element. It is reminiscent of Fregean numbers.
- The definition is predicative as the limit is not computed in $\mathcal{C}$ but in the possibly bigger $\mathcal{D}$. One can read $\mathcal{C}$ as the category of sets and $\mathcal{D}$ as the category of classes. The limit of sets possibly goes beyond sets.

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$
\begin{tikzcd}
F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\
F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}|
\end{tikzcd}
$$

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$\begin{tikzcd} F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\ F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}| \end{tikzcd}$$

making the limit cone into F-morphisms.

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$\begin{tikzcd}
F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\
F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}|
\end{tikzcd}$$

making the limit cone into F-morphisms. This F-structure is unique in this property.

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$\begin{tikzcd} F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\ F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}| \end{tikzcd}$$

  making the limit cone into F-morphisms. This F-structure is unique in this property.
- (Absolute case) If $\mathcal{C} = \mathcal{D}$ is finitely complete, then:

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$\begin{tikzcd}
F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\
F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}|
\end{tikzcd}$$

  making the limit cone into F-morphisms. This F-structure is unique in this property.

- (Absolute case) If $\mathcal{C} = \mathcal{D}$ is finitely complete, then:

$$F\text{-scheme} = \text{Initial } F\text{-algebra}$$

The Algebraic Structure on F-schemes

- Using the universality of limit, there is a canonical F-algebra structure on I:

$$\begin{tikzcd} F(I) \arrow[r, dotted, "a_I"] \arrow[d, "F(r_{\mathfrak{A}})"'] & I \arrow[d, "r_{\mathfrak{A}}"] \\ F(|\mathfrak{A}|) \arrow[r, "a_{\mathfrak{A}}"'] & |\mathfrak{A}| \end{tikzcd}$$

  making the limit cone into F-morphisms. This F-structure is unique in this property.

- (Absolute case) If $\mathcal{C} = \mathcal{D}$ is finitely complete, then:

$$F\text{-scheme} = \text{Initial } F\text{-algebra}$$

  The canonical F-algebra structure on the F-scheme is the initial F-algebra and the underlying object of any initial F-algebra is an F-scheme.

Something is Missing!

I was explaining these ideas in their embryonic form to Thierry Coquand when we both were at the Hausdorff institute (2018). He listened and then came up with an objection:

Something is Missing!

I was explaining these ideas in their embryonic form to Thierry Coquand when we both were at the Hausdorff institute (2018). He listened and then came up with an objection:

Your theory must see the difference between the real problematic infinitary F-algebras such as the F-algebra of natural numbers ($F(X) = 1 + X$) and the tame finitary ones such as the set with two elements ($F(X) = 1 + 1$)!

Something is Missing!

I was explaining these ideas in their embryonic form to Thierry Coquand when we both were at the Hausdorff institute (2018). He listened and then came up with an objection:

> Your theory must see the difference between the real problematic infinitary F-algebras such as the F-algebra of natural numbers ($F(X) = 1 + X$) and the tame finitary ones such as the set with two elements ($F(X) = 1 + 1$)!

Then, I kept the objection somewhere in my mind and put the problem aside. Recently, I picked it up again and then I saw the way to differentiate between the two!

Something is Missing!

I was explaining these ideas in their embryonic form to Thierry Coquand when we both were at the Hausdorff institute (2018). He listened and then came up with an objection:

> Your theory must see the difference between the real problematic infinitary F-algebras such as the F-algebra of natural numbers ($F(X) = 1 + X$) and the tame finitary ones such as the set with two elements ($F(X) = 1 + 1$)!

Then, I kept the objection somewhere in my mind and put the problem aside. Recently, I picked it up again and then I saw the way to differentiate between the two! How to solve the issue?

Something is Missing!

I was explaining these ideas in their embryonic form to Thierry Coquand when we both were at the Hausdorff institute (2018). He listened and then came up with an objection:

> Your theory must see the difference between the real problematic infinitary F-algebras such as the F-algebra of natural numbers ($F(X) = 1 + X$) and the tame finitary ones such as the set with two elements ($F(X) = 1 + 1$)!

Then, I kept the objection somewhere in my mind and put the problem aside. Recently, I picked it up again and then I saw the way to differentiate between the two! How to solve the issue?

An Idea

The limit is of course out of reach but it should not be too far! It must be approximable.

Approximability

- Defining a set as the "least" element in a huge family is impredicative.

Approximability

- Defining a set as the "least" element in a huge family is impredicative.
- However, one can still stratify the whole family into a directed family of small subfamilies (e.g. finite) for each of which we can accept the existence of the least element.

Approximability

- Defining a set as the "least" element in a huge family is impredicative.
- However, one can still stratify the whole family into a directed family of small subfamilies (e.g. finite) for each of which we can accept the existence of the least element.
- The initial objects in these families then approximate the limit better and better.

Approximability

- Defining a set as the "least" element in a huge family is impredicative.
- However, one can still stratify the whole family into a directed family of small subfamilies (e.g. finite) for each of which we can accept the existence of the least element.
- The initial objects in these families then approximate the limit better and better.
- Having these approximations, the finitary initial F-algebras become already reachable while the inifinitary ones remain out of the reach.

Approximability

- Defining a set as the "least" element in a huge family is impredicative.
- However, one can still stratify the whole family into a directed family of small subfamilies (e.g. finite) for each of which we can accept the existence of the least element.
- The initial objects in these families then approximate the limit better and better.
- Having these approximations, the finitary initial F-algebras become already reachable while the inifinitary ones remain out of the reach.

To formalize such subfamilies we need the following:

Definition

Let $\mathcal{C}$ be a category. A family $\{\mathcal{S}_i\}_{i \in I}$ of the families of $\mathcal{C}$-maps is called directed if for any $i, j \in I$, there exits $k \in I$ such that $\mathcal{S}_i$ and $\mathcal{S}_j$ are subsets of $\mathcal{S}_k$. It is called covering if any map of $\mathcal{C}$ is in one of the $\mathcal{S}_i$'s.

Predicative F-schemes

For any family $\mathcal{S}$ of maps closed under F, by an F-algebra over $\mathcal{S}$, we mean an F-algebra whose map is in $\mathcal{S}$.

Predicative F-schemes

For any family $\mathcal{S}$ of maps closed under F, by an F-algebra over $\mathcal{S}$, we mean an F-algebra whose map is in $\mathcal{S}$.

The F-algebra $\mathfrak{J}$ over $\mathcal{S}$ is called initial in $\mathrm{Alg}_F(\mathcal{S})$ if for any F-algebra $\mathfrak{A}$ over $\mathcal{S}$, there is an F-morphism $f : \mathfrak{J} \to \mathfrak{A}$ in $\mathcal{S}$ and it is unique among any such F-morphism constructed by a composition of the maps in $\mathcal{S}$.

Predicative F-schemes

For any family $\mathcal{S}$ of maps closed under F, by an F-algebra over $\mathcal{S}$, we mean an F-algebra whose map is in $\mathcal{S}$.

The F-algebra $\mathfrak{I}$ over $\mathcal{S}$ is called initial in $\mathrm{Alg}_F(\mathcal{S})$ if for any F-algebra $\mathfrak{A}$ over $\mathcal{S}$, there is an F-morphism $f : \mathfrak{I} \to \mathfrak{A}$ in $\mathcal{S}$ and it is unique among any such F-morphism constructed by a composition of the maps in $\mathcal{S}$.

Definition

The F-scheme of $\mathcal{C}$ in $\mathcal{D}$ is called predicative if there exists a directed covering family $\{\mathcal{S}_i\}_{i \in I}$ of families of $\mathcal{C}$-maps such that each $\mathrm{Alg}_F(\mathcal{S}_i)$ has an initial object.

Predicative F-schemes

For any family $\mathcal{S}$ of maps closed under F, by an F-algebra over $\mathcal{S}$, we mean an F-algebra whose map is in $\mathcal{S}$.

The F-algebra $\mathfrak{I}$ over $\mathcal{S}$ is called initial in $\mathrm{Alg}_F(\mathcal{S})$ if for any F-algebra $\mathfrak{A}$ over $\mathcal{S}$, there is an F-morphism $f : \mathfrak{I} \to \mathfrak{A}$ in $\mathcal{S}$ and it is unique among any such F-morphism constructed by a composition of the maps in $\mathcal{S}$.

Definition

The F-scheme of $\mathcal{C}$ in $\mathcal{D}$ is called predicative if there exists a directed covering family $\{\mathcal{S}_i\}_{i \in I}$ of families of $\mathcal{C}$-maps such that each $\mathrm{Alg}_F(\mathcal{S}_i)$ has an initial object.

Now, we have completed our definition of predicative F-schemes. How to use them for computation?

Representable Functions

We saw that any $F_{\mathbb{N}}$-scheme N has a zero element and a successor. Therefore, it is possible to represent:

Representable Functions

We saw that any $F_{\mathbb{N}}$-scheme N has a zero element and a successor. Therefore, it is possible to represent:

- the natural number $n \in \mathbb{N}$ by the map $\bar{n} = s_N s_N \ldots s_N 0_N : 1 \to N$, where n is the number of s_N's, and

Representable Functions

We saw that any $F_{\mathbb{N}}$-scheme N has a zero element and a successor. Therefore, it is possible to represent:

- the natural number $n \in \mathbb{N}$ by the map $\bar{n} = s_N s_N \ldots s_N 0_N : 1 \to N$, where n is the number of s_N's, and
- a function $f : \mathbb{N}^k \to \mathbb{N}$ by a map $F : N^k \to N$ if for any $(n_1, \ldots, n_k) \in \mathbb{N}^k$, we have:

$$\begin{tikzcd} 1 \arrow[d, "(\overline{n_1},\overline{n_2},\ldots,\overline{n_k})"'] \arrow[dr, "\overline{f(n_1,n_2,\ldots,n_k)}"] & \\ N^k \arrow[r, "F"'] & N \end{tikzcd}$$

Representable Functions

We saw that any $F_{\mathbb{N}}$-scheme N has a zero element and a successor. Therefore, it is possible to represent:

- the natural number $n \in \mathbb{N}$ by the map $\bar{n} = s_N s_N \ldots s_N 0_N : 1 \to N$, where n is the number of s_N's, and
- a function $f : \mathbb{N}^k \to \mathbb{N}$ by a map $F : N^k \to N$ if for any $(n_1, \ldots, n_k) \in \mathbb{N}^k$, we have:

$$\begin{tikzcd} 1 \arrow[d, "(\overline{n_1},\overline{n_2},\ldots,\overline{n_k})"'] \arrow[dr, "\overline{f(n_1,n_2,\ldots,n_k)}"] & \\ N^k \arrow[r, "F"'] & N \end{tikzcd}$$

We can have a similar representation for binary strings by $F_{\mathbb{W}}$-schemes.

Characterization of Representable Functions

What functions are representable by all $F_{\mathbb{N}}$-schemes or $F_{\mathbb{W}}$-schemes?

Characterization of Representable Functions

What functions are representable by all $F_{\mathbb{N}}$-schemes or $F_{\mathbb{W}}$-schemes?

Theorem

- *A function $f : \mathbb{N}^k \to \mathbb{N}$ is representable by all $F_{\mathbb{N}}$-schemes iff it is linear space computable.*

Characterization of Representable Functions

What functions are representable by all $F_{\mathbb{N}}$-schemes or $F_{\mathbb{W}}$-schemes?

Theorem

- *A function $f : \mathbb{N}^k \to \mathbb{N}$ is representable by all $F_{\mathbb{N}}$-schemes iff it is linear space computable.*
- *A function $f : \mathbb{W}^k \to \mathbb{W}$ is representable by all $F_{\mathbb{W}}$-schemes iff it is polynomial time computable.*

Characterization of Representable Functions

What functions are representable by all $F_{\mathbb{N}}$-schemes or $F_{\mathbb{W}}$-schemes?

Theorem

- *A function $f : \mathbb{N}^k \to \mathbb{N}$ is representable by all $F_{\mathbb{N}}$-schemes iff it is linear space computable.*
- *A function $f : \mathbb{W}^k \to \mathbb{W}$ is representable by all $F_{\mathbb{W}}$-schemes iff it is polynomial time computable.*

Caveat!

To have the previous theorem, one needs to add parameters everywhere in the definitions! We omit them for the sake of clearer presentation.

The Idea behind the Proof

Unfortunately, I don't have time to go into the details. However, the main idea is:

The Idea behind the Proof

Unfortunately, I don't have time to go into the details. However, the main idea is:

- Being a limit allows for a restricted recursion exactly in the amount needed for low-complexity functions.

The Idea behind the Proof

Unfortunately, I don't have time to go into the details. However, the main idea is:

- Being a limit allows for a restricted recursion exactly in the amount needed for low-complexity functions.
- However, the fact that the limit is outside $\mathcal{C}$ disallows arbitrary iterations we usually have for initial F-algebras.

Comparison with the Literature

To capture low-complexity functions, people usually use:

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,
- linear-style restrictions as in light affine set theory.

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,
- linear-style restrictions as in light affine set theory.

We believe that understanding feasible mathematics calls for:

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,
- linear-style restrictions as in light affine set theory.

We believe that understanding feasible mathematics calls for:

- a genuinely alternative yet rich notion of an inductive object,

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,
- linear-style restrictions as in light affine set theory.

We believe that understanding feasible mathematics calls for:

- a genuinely alternative yet rich notion of an inductive object,
- such a notion must be defined in a universal way so that we can claim we really understand the notion,

Comparison with the Literature

To capture low-complexity functions, people usually use:

- syntactic restrictions as in Leivant's and Cook-Bellantoni's predicative recursion or using monads in Hofmann's categorical characterizations,
- linear-style restrictions as in light affine set theory.

We believe that understanding feasible mathematics calls for:

- a genuinely alternative yet rich notion of an inductive object,
- such a notion must be defined in a universal way so that we can claim we really understand the notion,
- it must be closer to the usual practice of mathematics to make it easy to use.

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	Analytic

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	**Analytic**
Russian Constructivism	Computable Mathematics

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	**Analytic**
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	**Analytic**
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry
Martin Löf type theory	Homotopy theory

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	**Analytic**
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry
Martin Löf type theory	Homotopy theory

Finding a categorical characterization of low-complexity computation:

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	**Analytic**
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry
Martin Löf type theory	Homotopy theory

Finding a categorical characterization of low-complexity computation:

- unearths the internal structure of the low-complexity computation,

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	Analytic
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry
Martin Löf type theory	Homotopy theory

Finding a categorical characterization of low-complexity computation:

- unearths the internal structure of the low-complexity computation,
- absolute predicativism can be our light to navigate,

The Tip of the Iceberg

What we said, we believe, is the tip of the following iceberg:

The Main Thesis

Absolutly Predicative Mathematics = Feasible Mathematics

Synthetic	Analytic
Russian Constructivism	Computable Mathematics
Brouwerian Constructivism	Geometry
Martin Löf type theory	Homotopy theory

Finding a categorical characterization of low-complexity computation:

- unearths the internal structure of the low-complexity computation,
- absolute predicativism can be our light to navigate,
- helps to provide more models to prove independence results.

Thank you for your attention!