Coherence, conjectures, and congruential functions

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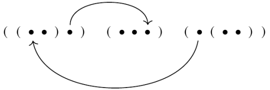
Structure Meets Power

Paris, le 4 Juillet 1932(+90)



Au fond de l'Inconnu pour trouver du nouveau!

"In his notebook dated July 1, 1932, he [Lothar Collatz] *considered the function*

$$n \mapsto \begin{cases} \frac{2}{3}n & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{4}{3}n - \frac{1}{3} & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{4}{3}n + \frac{1}{3} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

He posed the problem of whether the cycle containing 8 *is finite or infinite. I will call this the* **Original Collatz Problem**. *His original question has never been answered."*

The $3x + 1$ problem & its generalisations
– Jeffrey Lagarias (1985)

The bijection from the OCP is a *congruential function*

From J. Conway's "Unpredictable Iterations" paper (1972), this problem –might– be undecidable, but could never be proved to be so.

Thanks to J. Lagarias for references & anecdotes on the OCP !

I want to play a game!

Two players – **Alice** and **Bob** – play against a **Dealer** with an infinite deck of cards.

The game is based around shuffling and dealing packs of cards.

- **Fair deals** – passing cards to each player, in turn.
- **Riffle (or Fano) shuffles** — perfect interleavings of decks of cards.

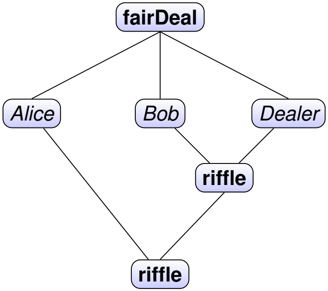
The nature of my game

To start the game : Alice and Bob mark a single card (number 8).

Step 1 The **Dealer** shares out the cards to all players, including himself.

Step 2 **Bob** passes his hand of cards to the **Dealer**, who shuffles it together with his own hand of cards.

Step 3 **Alice** does the same, leaving the **Dealer** holding all the cards.



In each round, Steps 1.-3. are repeated :

They win when **their card** returns to its **original position** in the Dealer's hand.

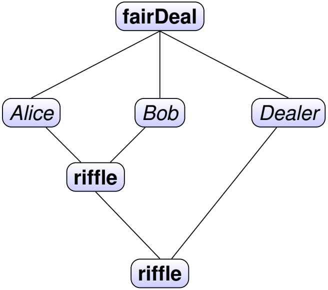
The other way to play

To start the game : Alice and Bob mark a single card (number 7).

Step 1 The **Dealer** shares out the cards to all players, including himself.

Step 2 **Alice** passes her hand of cards to **Bob**, who shuffles it together with his own hand of cards.

Step 3 **Bob** passes the result to the Dealer, who shuffles it together with his hand.

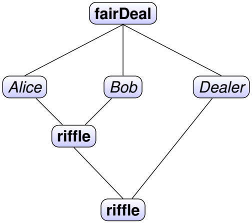


The two games can **never** be the same!

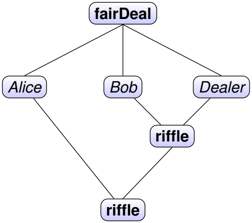
– a corollary of, *"Coherence & Strictification for Self-Similarity"*
Journal of Homotopy & Related Structures (PMH 2016)

The two paths you can go by

The **left-associated Shuffle Game**



The **right-associated Shuffle Game**



The **left Collatz bijection**

$$\gamma_L(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \ (mod\ 3) \\ \frac{4n+2}{3} & n \equiv 1 \ (mod\ 3) \\ \frac{2n-1}{3} & n \equiv 2 \ (mod\ 3) \end{cases}$$

The **right Collatz bijection**

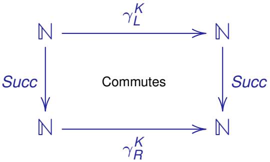
$$\gamma_r(n) = \begin{cases} \frac{2}{3}n & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{4n-1}{3} & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{4n-1}{3} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

The two games are simply *shifted versions of each other*

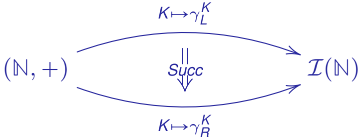
$$1 + \gamma_L^K(n) = \gamma_R^K(n+1) \qquad \forall\ K \in \mathbb{N}$$

Natural transformations between left- and right- associativity

For all $K \in \mathbb{N}$ we have a commuting diagram of **partial injective functions** :



The successor function $Succ \in \mathcal{I}(\mathbb{N})$ is the **unique component** of a **natural transformation** between monoid homomorphisms



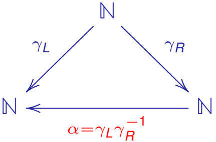
Using the **generalised inverse** of the successor,

$$\gamma_L^K = succ^{-1}.\gamma_R^K.succ \quad \text{but} \quad \gamma_R^K \neq succ.\gamma_L^K.succ^{-1}$$

A more ‘traditional’ approach to associativity

Define the **associator** $\alpha \in \mathcal{I}(\mathbb{N})$ to be the bijection that maps :

1. The result of the **right** shuffle game, to **2.** The result of the **left** shuffle game.



$$\alpha(n) = \gamma_L \gamma_R^{-1}(n) = \begin{cases} 2n & n \equiv 0 \ (mod\ 2) \\ n+1 & n \equiv 4 \ (mod\ 4) \\ \frac{n-1}{2} & n \equiv 3 \ (mod\ 4) \end{cases}$$

The **associator** is a **commutator** :)

Using the connection with the successor function & its generalised inverse :

$$\alpha = \gamma_L \gamma_R^{-1} = Succ^{-1}.\gamma_R.Succ.\gamma_R^{-1} = \left[Succ, \gamma_R^{-1} \right]$$

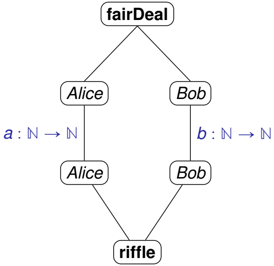
Remark : Characterising finite orbits under the associator is *trivial*.

$$\alpha(0) = 0 \ , \ \ \alpha^K(n) \neq n \quad \forall K \ , \ n > 0$$

Writing the Dealer out of the game

Girard's model of **multiplicative conjunction**, from Geometry of Interaction (I) and (II)

1. Cards are dealt out to **Alice** and **Bob**.
2. They both apply their favourite (partial?) permutation :
   - Alice applies $a : \mathbb{N} \rightarrow \mathbb{N}$,
   - Bob applies $b : \mathbb{N} \rightarrow \mathbb{N}$.
3. Their (permuted) cards are shuffled back together.



As a homomorphism $(\_ \star \_) \ : \ \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$

$$(a \star b)(n) \ = \ \begin{cases} 2.a\left(\frac{n}{2}\right) & n \text{ even}, \\ 2.b\left(\frac{n-1}{2}\right) + 1 & n \text{ odd}. \end{cases}$$

Alice and Bob in conjunction

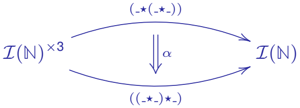
Girard's Conjunction $(\_ \star \_)$ is a (semi-monoidal) categorical tensor on $\mathcal{I}(\mathbb{N})$

- It is associative **up to (fixed) isomorphism**

$$\alpha.(a \star (b \star c)) \ = \ ((a \star b) \star c).\alpha \quad \forall\, a, b, c \in \mathcal{I}(\mathbb{N})$$

  i.e. the associator $\alpha = \left[Succ, \gamma_R^{-1} \right]$.

- This is the **unique component** of a natural isomorphism :



- MacLane's **pentagon condition** is satisfied.

$$\alpha^2 \ = \ (\alpha \star Id)\alpha(Id \star \alpha)$$

The group of canonical isomorphisms

As a corollary of :

"The Structure Group for the Associativity Identity"
— Patrick Dehornoy (1996)

the **subgroup** of $\mathcal{I}(\mathbb{N})$ generated by the **bijections**

$$\begin{array}{rcl} X_0 & = & \alpha \\ X_1 & = & (Id \star \alpha) \\ X_2 & = & (Id \star (Id \star \alpha)) \\ X_3 & = & (Id \star (Id \star (Id \star \alpha))) \\ & \vdots & \end{array}$$

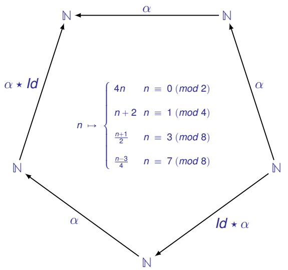
is isomorphic to **Thompson's group** $\mathcal{F} = \langle X_i \ : \ X_i^{-1} X_j X_i = X_{j+1} \quad \forall i < j \in \mathbb{N} \rangle$.

A *minimal* generating set is :

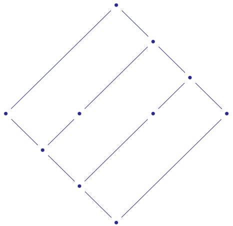
$$X_0(n) = \begin{cases} 2n & n \equiv 0 \ (mod\ 2) \\ n+1 & n \equiv 1 \ (mod\ 4) \\ \frac{n-1}{2} & n \equiv 3 \ (mod\ 4) \end{cases} \qquad X_1(n) = \begin{cases} n & n \equiv 0 \ (mod\ 2) \\ 2n-1 & n \equiv 1 \ (mod\ 4) \\ n+2 & n \equiv 3 \ (mod\ 8) \\ \frac{n-1}{2} & n \equiv 7 \ (mod\ 8) \end{cases}$$

MacLane's Pentagon for Girard's conjunction

A commuting diagram



A series of shuffles & deals

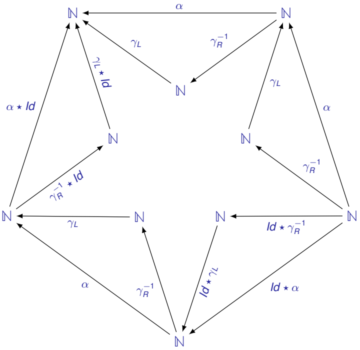


"Elementary arithmetic" proofs, for MacLane's pentagon and hexagon

"Modular arithmetic identities from untyped categorical coherence", Reversible Computing, Springer L.N.C.S. *(PMH - 2013)*

Alice and Bob split the associator

Let's add in the decomposition of the associator $\alpha = \gamma_L \gamma_R^{-1}$ to MacLane's pentagon!

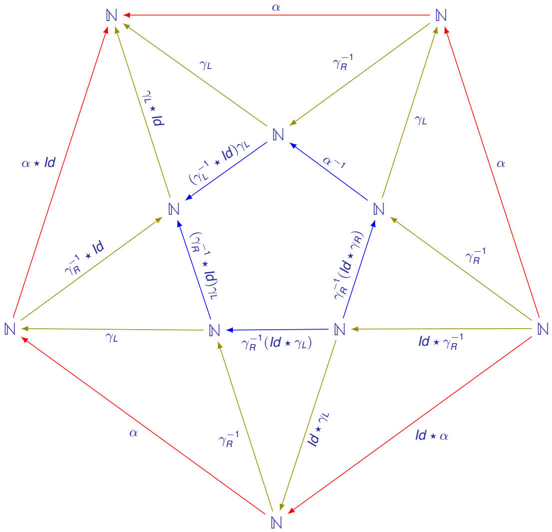


Where :

- $(\_ \star \_)$ is Girard's conjunction,
- $\gamma_R : \mathbb{N} \rightarrow \mathbb{N}$ is the operator from the original Collatz conjecture,
- $\gamma_L = succ^{-1}.\gamma_R.succ$ is another way of expressing Collatz's conjecture.
- $\alpha = [succ, \gamma_R^{-1}] = \gamma_L \gamma_R^{-1}$ is the associator for $(\_ \star \_)$.

The temptation to complete the inner pentagon is overwhelming!

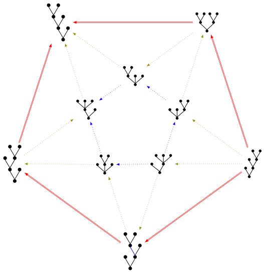
A commuting diagram :



A Convergent Series?

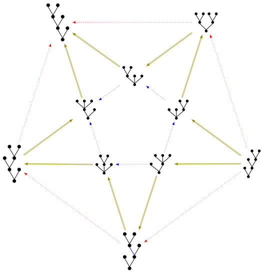
MacLane's pentagon is the 1-skeleton of Stasheff's associahedron $\mathcal{K}_4$;
we understand the rest of the pentagram in similar terms.

Mapping between Vertices



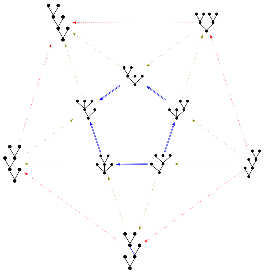
The associator for Girard's $(\_ \star \_)$

Mapping between Edges & Vertices



Collatz's bijection(s)

Mapping between Edges



Girard-Collatz composites

These are (unique components of) natural transformations in a functor category

A and B, and the Dealer makes three

Girard's conjunction : $(\_ \star \_) : \mathcal{I}(\mathbb{N})^{\times 2} \hookrightarrow \mathcal{I}(\mathbb{N})$, defined by

$$(a \star b)(n) = \begin{cases} 2.a\left(\frac{n}{2}\right) & n \equiv 0 \ (mod\ 2) \\ 2.b\left(\frac{n-1}{2}\right) + 1 & n \equiv 1 \ (mod\ 2) \end{cases}$$

The three-fold conjunction : We define $(\_ \star \_ \star \_) : \mathcal{I}(\mathbb{N})^{\times 3} \hookrightarrow \mathcal{I}(\mathbb{N})$, by

$$(a \star b \star d)(n) = \begin{cases} 3.a\left(\frac{n}{3}\right) & n \equiv 0 \ (mod\ 3) \\ 3.b\left(\frac{n-1}{3}\right) + 1 & n \equiv 1 \ (mod\ 3) \\ 3.d\left(\frac{n-2}{3}\right) + 2 & n \equiv 2 \ (mod\ 3) \end{cases}$$

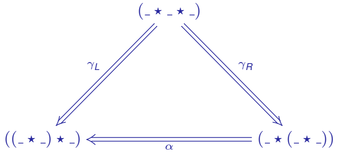
Which **natural transformations** relate the following

$$(\_ \star \_ \star \_) \ , \ (\_ \star (\_ \star \_)) \ , \ ((\_ \star \_) \star \_) \ : \ \mathcal{I}(\mathbb{N})^{\times 3} \hookrightarrow \mathcal{I}(\mathbb{N})$$

injective homomorphisms?

A familiar game!

In the category of homomorphisms / natural transformations :



The natural interpretation :

The associator / its inverse : left- and right- associated re-bracketing

The Collatz bijections / their inverses : deleting / inserting brackets

We may view **re-bracketing** as **deleting** then **re-inserting** brackets.

Conveniently, the category generated by $(\_ \star \_), (\_ \star \_ \star \_)$, along with these natural isomorphisms between them, is also **posetal**[1].

[1] A special case of a more general result ...

References & Acknowledgements

https://arXiv.org/abs/2202.04443v1 From a conjecture of Collatz to Thompson's group $\mathcal{F}$, via a conjunction of Girard.

https://arxiv.org/abs/2206.07412v2 The inverse semigroup theory of elementary arithmetic.

Many thanks to :

Matt Brin (Binghampton), for history, theory, & references on Thompson groups & their connection with coherence.

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