# A Categorical Approach to Descriptive Complexity Theory

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## Boolean Categories

| | |
|---|---|
| Boolean toposes | higher-order theories |
| Boolean pretoposes | first-order theories |
| Boolean lextensive categories | "quantifier-free" theories |

- Boolean (lextensive) category [Carboni, Lack, Walters 1993]:
  - finite products (the structure below in fact implies all finite limits);
  - finite coproducts;
  - $1 + 1$ is disjoint, pullback-stable and the subobject classifier.

- Logical functor: functor preserving fin. prods. and fin. coprods.

- Examples (small):
  - $\mathcal{F}$: (skeleton of) finite sets and functions;
  - $\mathcal{F}_\omega$: (skeleton of) countable sets and functions;
  - syntactic categories of Boolean theories (next slide).

## Boolean Categories of Finite Presentation

- Finite Boolean theory $\mathbb{T} := (\mathrm{Sort}, \mathrm{Rel}, \mathrm{Ax})$:
  - $\mathrm{Sort}$ finite set of *sorts*;
  - $\mathrm{Rel}$ finite set of *relation symbols*, $R \rightarrowtail A_1, \ldots, A_k$, with $A_i$ sorts;
  - $\mathrm{Ax}$ finite set of *axioms*, quantifier-free (except for provably unique $\exists$).

  (So a Boolean theory is a multisorted, relational FO theory with equality, with closed axioms of the form $\forall \vec{x}.\varphi$ with $\varphi$ quantifier-free except for provably unique $\exists$).

- $\mathcal{F}[\mathbb{T}]$: the cat of definable sets and functions in $\mathbb{T}$.
  - $\mathcal{F} = \mathcal{F}[\mathbb{E}]$ where $\mathbb{E}$ is the empty theory.
  - The obj. of $\mathcal{F}[N; E \rightarrowtail N^2]$ are "polynomials" on $N$, $E$ and $\overline{E}$.
  - $\mathcal{F}_\omega$ is not of finite presentation.

- $\mathcal{B}ool\mathcal{C}at_{\mathrm{fp}} :=$ fin. pres. Bool cats and logical functors modulo iso.

- Analogy with $\mathbb{Q}$-algebras: $\mathcal{F}[\mathrm{Sort}; \mathrm{Rel}]/\langle \mathrm{Ax} \rangle$ is like $\mathbb{Q}[\mathrm{Var}]/\langle \mathrm{Poly} \rangle$.

## Data Presheaves

- Define a data presheaf as a functor $\mathcal{B}ool\mathcal{C}at_{\mathrm{fp}} \to \mathbf{Set}$.
  - We write $\mathrm{Spec}\,\mathcal{B} := \mathcal{B}ool\mathcal{C}at_{\mathrm{fp}}(\mathcal{B}, -)$ (a data specification).
  - Morphism $\mathrm{Spec}\,\mathcal{B} \to \mathrm{Spec}\,\mathcal{A}$ = logical functor $\mathcal{A} \to \mathcal{B}$ (mod. iso).
  - The (small) category $\mathcal{D}ata$ of data specs is lextensive.

- Global section functor $\Gamma : \mathbf{DataPSh} \to \mathbf{Set}$: $X \mapsto X(\mathcal{F})$
  - $\Gamma(\mathrm{Spec}\,\mathcal{F}[\mathbb{T}]) = \{$logical $\mathcal{F}[\mathbb{T}] \to \mathcal{F}\}$ = finite models of $\mathbb{T}$.
  - If $f : X \to \mathrm{Spec}\,\mathcal{F}[\mathbb{T}]$, $\mathrm{im}\,\Gamma(f) \subseteq \Gamma(\mathrm{Spec}\,\mathcal{F}[\mathbb{T}])$ is a Boolean query.
  - Morphism $h : \mathrm{Spec}\,\mathcal{F}[\mathbb{S}] \to \mathrm{Spec}\,\mathcal{F}[\mathbb{T}] \approx$ quantifier-free query from the fin. models of $\mathbb{S}$ to the fin. models of $\mathbb{T}$.
  - Pullback $h^* f : X' \to \mathrm{Spec}\,\mathcal{F}[\mathbb{S}]$ = quantifier-free reduction (parsimonious).

- $\mathbb{S}\mathrm{tr}$: 1 sort, total order, predicate isOne. $\mathbb{S}\mathrm{tr}_{+\times} := \mathbb{S}\mathrm{tr} +$ arithmetic. $\mathrm{Spec}\,\mathcal{F}[\mathbb{S}\mathrm{tr}_{+\times}] \to \mathrm{Spec}\,\mathcal{F}[\mathbb{S}\mathrm{tr}]$ = uniform $\mathrm{NC}^0$ function.

## Computability and Complexity

- If $f : \mathcal{F}[\mathbb{T}] \to \mathcal{B}$ with $\mathcal{B}$ fin. pres., then $\mathcal{B} \cong \mathcal{F}[\mathbb{T}_f]$ s.t. $\mathrm{Sort}(\mathbb{T}) \subseteq \mathrm{Sort}(\mathbb{T}_f)$, $\mathrm{Rel}(\mathbb{T}) \subseteq \mathrm{Rel}(\mathbb{T}_f)$, $\mathrm{Ax}(\mathbb{T}) \subseteq \mathrm{Ax}(\mathbb{T}_f)$ and $f = \mathrm{iso} \circ \mathrm{inclusion}$.

- Types of morphism $\mathrm{Spec}\,\mathcal{B} \to \mathrm{Spec}\,\mathcal{F}[\mathbb{T}]$ according to the dual $f$:
  - relational: $\mathrm{Sort}(\mathbb{T}_f) = \mathrm{Sort}(\mathbb{T})$;
  - Horn: relational + constraints on $\mathrm{Ax}(\mathbb{T}_f) \setminus \mathrm{Ax}(\mathbb{T})$;
  - Krom: relational + other constraints on $\mathrm{Ax}(\mathbb{T}_f) \setminus \mathrm{Ax}(\mathbb{T})$.

**Theorem.** *$A \subseteq \{0,1\}^*$ is r.e. iff $\exists f : X \to \mathrm{Spec}\,\mathcal{F}[\mathbb{S}\mathrm{tr}]$ s.t. $A = \mathrm{im}\,\Gamma(f)$. Moreover:*

- $A \in$ NP iff $f$ is relational;
- $A \in$ P iff $f$ is Horn;
- $A \in$ NL iff $f : X \to \mathrm{Spec}\,\mathcal{F}[\mathbb{S}\mathrm{tr}_{+\times}]$ is Krom.

## Complete Problems

- Define $R, R_\bullet : \mathcal{B}ool\mathcal{C}at_{\mathrm{fp}} \to \mathbf{Set}$ by (on arrows, act by pullback):

$$R(\mathcal{B}) := \{\text{relational morphisms over } \operatorname{Spec}\mathcal{B}\}$$

$$R_\bullet(\mathcal{B}) := \{(f, s) \mid f \text{ rel. morphism over } \operatorname{Spec}\mathcal{B},\ f \circ s = \mathrm{id}\}$$

- Proj. $u : R_\bullet \to R$ = "universal" NP problem: $\forall$ rel. mor. $f : U \to \operatorname{Spec}\mathcal{F}[\mathbb{S}]$,

$$\begin{array}{ccc} U_{\lrcorner} & \longrightarrow & R_\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ \operatorname{Spec}\mathcal{F}[\mathbb{S}] & \xrightarrow{\exists !} & R \end{array}$$

$\Gamma(R) = \{\text{CNFs}\}$, $\Gamma(R_\bullet) = \{(\varphi, \sigma) \mid \sigma \models \varphi\}$ and $\operatorname{im}\Gamma(u) = \textsc{Sat}$.

- Can do the same with Horn ($H, H_\bullet$) and Krom ($K, K_\bullet$) morphism:
  - for $H$, $\operatorname{im}\Gamma(u) = \textsc{Horn Sat}$;
  - for $K$, $\operatorname{im}\Gamma(u) = \textsc{Krom Sat}$.

## Perspectives

- More complexity classes?
  - CSPs are immediate. Uniform $AC^0 = LH = FO$ seems easy.
  - L maybe. Don't know about PH or PSPACE.
  - In any case, is the "moduli space" of these classes meaningful?

- A "fibrational" view of (search) problems?

- Colimits of presheaves are bad. We need sheaves.

- Algebraic geometry with Boolean cats instead of comm. rings?
  - Bool cats are intriguingly similar to $\mathbb{Q}$-algebras which are integral domains.
  - Zarisky topology? Data schemes = locally representable sheaves?
    (*Categories of spaces built from local models* [Zhen Lin Low 2016]).
  - Caveat: likely, alg. geom. questions $\not\cong$ complexity theory questions!