Query Algorithms Based on Homomorphism Counts

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Joint work with Phokion G. Kolaitis

Homomorphism Counts

Let G and H be two graphs (**finite**, **undirected** and **simple**).

1. Homomorphism from G to H: A function $h : V(G) \to V(H)$ such that for all $u, v \in V(G)$:
$$\text{if} \quad (u, v) \in E(G), \quad \text{then} \quad (h(u), h(v)) \in E(H).$$

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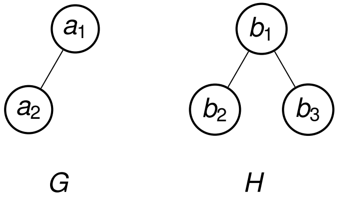
2. $\text{hom}(G, H)$: **number** of homomorphisms from G to H.

E.g., $\text{hom}(G, H) = 4$ for the graphs G and H below:



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Chen, Flum, Liu and Xun (2022) introduced the notions below:

1. $\mathcal{C}$ admits a k-non-adaptive left algorithm ($k \geq 1$) if there are k fixed graphs $F_1, F_2, \ldots, F_k$ such that for every G,
$$G \in \mathcal{C} \quad \text{iff} \quad (n_1, n_2, \ldots, n_k) \in X$$
where
   - $X \subseteq \mathbb{N}^k$ is decidable ($\mathbb{N}$: non-negative integers), and
   - $n_1 := \hom(F_1, G), n_2 := \hom(F_2, G), \ldots, n_k := \hom(F_k, G)$ are left homomorphism counts as queries made to the input G.

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2. $\mathcal{C}$ admits a k-adaptive left algorithm ($k \geq 1$) if the same holds except that $F_i = F_i(n_1, \ldots, n_{i-1})$ is a function of $n_1, \ldots, n_{i-1}$ for $2 \leq i \leq k$.

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3. "$\mathcal{C}$ admits a k-(non-)adaptive right algorithm" is analogous.

Non-Adaptive Left Algorithms

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The following classes admit a non-adaptive left algorithm (for some $k \geq 1$):

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but not the class of graphs containing an isolated node.

CSPs and Non-Adaptive Query Algorithms

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Theorem (Kolaitis-W. 2022):

For every H, the class $\mathrm{CSP}(H)$ admits a k-non-adaptive left algorithm for some $k \geq 1$ iff H contains no edge.

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- Trivial when H contains no edge: $k = 1$, take $F_1 = K_2$ (single-edge graph) and $X = \{0\}$. ($\hom(K_2, G) = 2 \times |E(G)|$.)

Proof of the Theorem

Let H contain an edge. Show that for every finite class $\mathcal{F}$ of graphs, there are $G_0 \in \mathrm{CSP}(H), G_1 \notin \mathrm{CSP}(H)$ with $\hom(F, G_0) = \hom(F, G_1)$ for $F \in \mathcal{F}$.

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Case 1: $\chi(H) = 2$. Then $\mathrm{CSP}(H) = \{G \mid G \text{ is 2-colorable}\}$.

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Theorem (Chen-Flum-Liu-Xun 2022):

For $n > 0$, two graphs $F_1 = F_1(n), F_2 = F_2(n)$ can be constructed such that for all G, H of size n:

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Three adaptive left queries $\hom(I_1, \cdot), \hom(F_1, \cdot), \hom(F_2, \cdot)$ suffice to determine, for all G and H, whether $G \cong H$.

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- $\mathrm{hom}(I_1, G) = |V(G)|$ (I_1: single-node graph)
- Optimal in number of queries made when only left homomorphism counts are allowed.

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Proof of the Theorem

Proof based on theorem by Chaudhuri and Vardi (1993):

$$G \cong H \qquad \text{iff} \qquad \hom(G, F) = \hom(H, F) \text{ for all } F.$$

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- Make D-ary representation of $\hom(G, F_0(n))$ contain all $\hom(G, A_j)$ as certain digits.

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$$\hom(G, F_0) = \sum_{e \in \mathcal{E}^+_{\leq n}} \left(\sum_{\substack{1 \leq j_1, \ldots, j_r \leq s \\ e_{j_1} + \cdots + e_{j_r} = e}} \prod_{k=1}^{r} \hom(G_k, A_{j_k}) \right) \times D^e.$$

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2. D^{re_j}-digit of D-ary representation of $\mathrm{hom}(G, F_0)$ is $\mathrm{hom}(G, A_j)$ for all $1 \leq j \leq s$.

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3. Allowing the number of queries to depend on input graph G