Recent Advances in Homomorphism Indistinguishability

Tim Seppelt

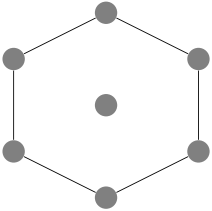
Joint work with Gaurav Rattan

Structure Meets Power 2022

Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic
UnRAVeL
RWTH AACHEN UNIVERSITY

DFG Deutsche Forschungsgemeinschaft
German Research Foundation

Homomorphism Indistinguishability



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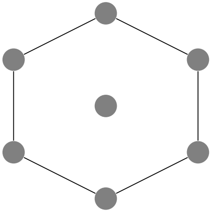

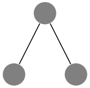

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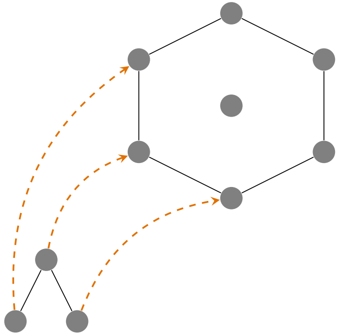

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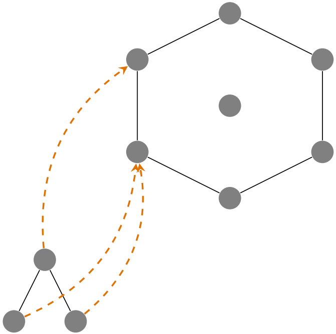

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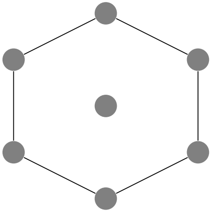

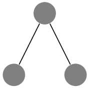

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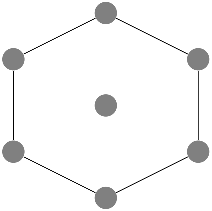

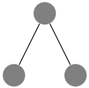

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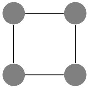

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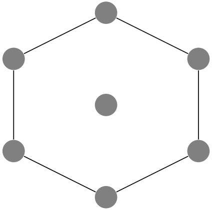
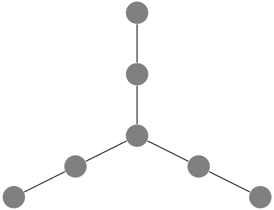
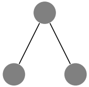
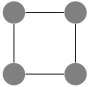
		
	24	24
	36	36

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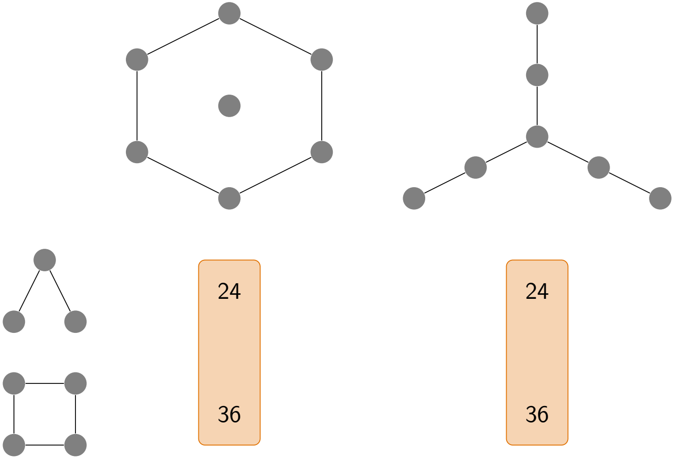

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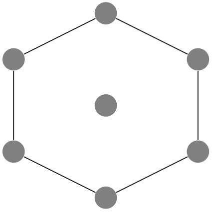
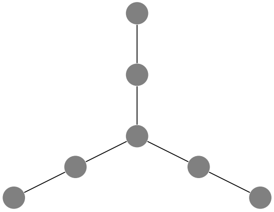
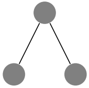
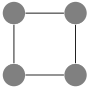
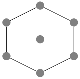
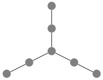








(path)	24	24
(square)	36	36

The graphs  and  are **homomorphism indistinguishable** over $\{$  ,  $\}$.

Why?

Logical Equivalences ⟵ **Homomorphism Indistinguishability** ⟶ **Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Why?

Logical Equivalences	←	**Homomorphism Indistinguishability**	→	**Matrix Equations** X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$
FO with counting C_2	←	Trees	→	Fractional Isomorphism

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		Planar graphs	⟶	Quantum Isomorphism

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FO with counting $\mathsf{C}_k \cap \mathsf{C}^q$	⟵	k-pebble forest covers of depth q	⟶	Novel System

Techniques

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Techniques

Logical Equivalences $\longleftarrow$ **Homomorphism Indistinguishability** $\longrightarrow$ Matrix Equations X s.t. $X\boldsymbol{A}_G = \boldsymbol{A}_H X$

Category Theory

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathfrak{C}A \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mathfrak{C}\alpha} \\ \mathfrak{C}A & \xrightarrow{\delta_A} & \mathfrak{C}\mathfrak{C}A \end{array}$$

Dawar et al. (2021); Montacute and Shah (2021); Abramsky et al. (2022); Reggio (2021); etc.

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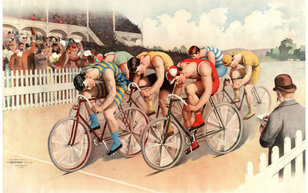
Algebra and Representation Theory

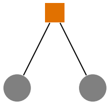


Mančinska and Roberson (2020); Grohe et al. (2022); Rattan and Seppelt (2022); etc.

The Recipe for Matrix Equations

1. Construct family $\mathcal{F}$ of labelled graphs
2. Define suitable operations
3. Prove that $\mathcal{F}$ is finitely generated under operations
4. Define representation and recover system of equations

Labelled Graphs and Homomorphism Tensors



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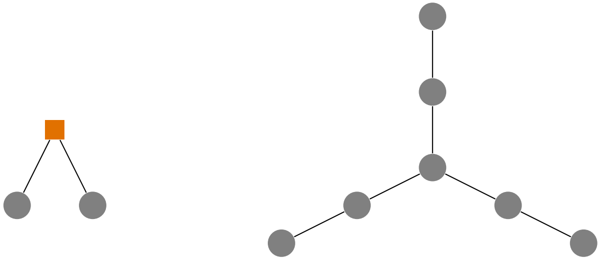

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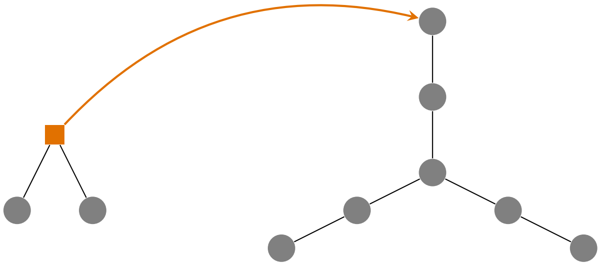

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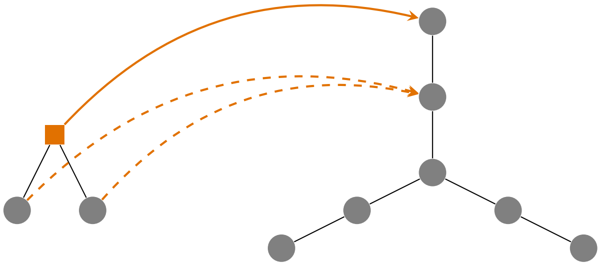

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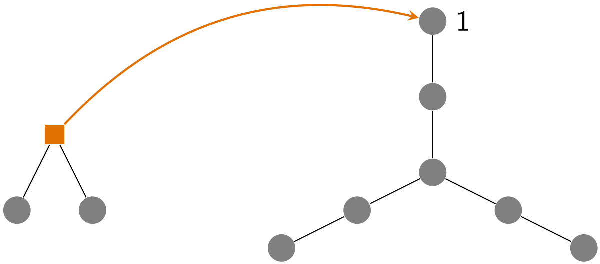

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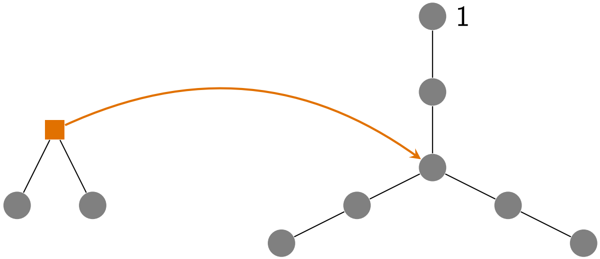

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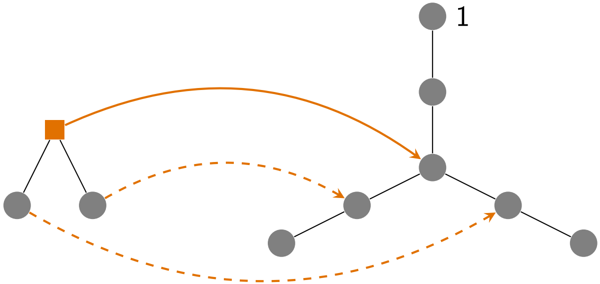

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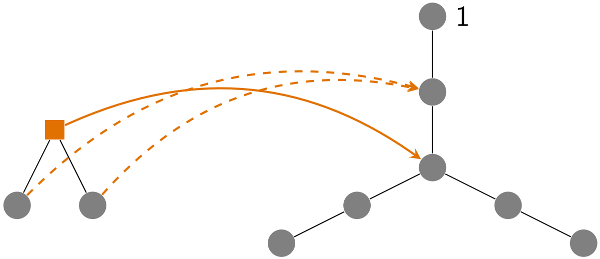

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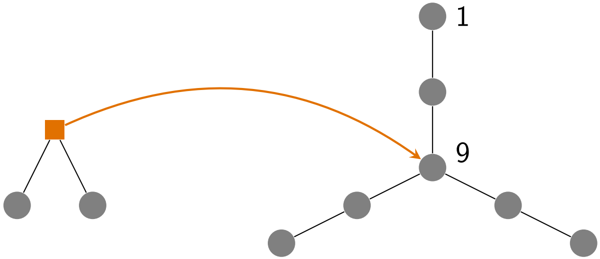
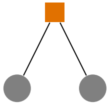
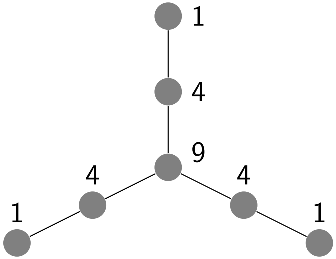


Labelled Graphs and Homomorphism Tensors

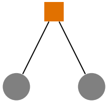
$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



$\mapsto$



Combinatorial and Algebraic Operations: Unlabelling and Sum-of-Entries



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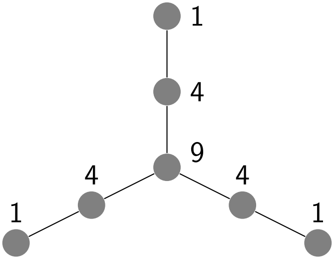

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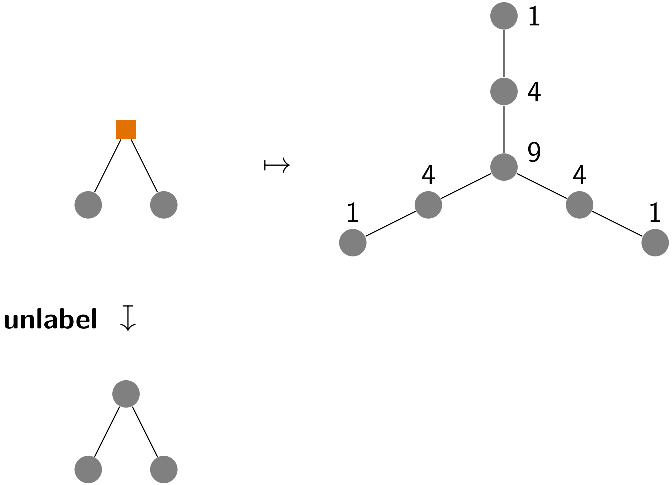

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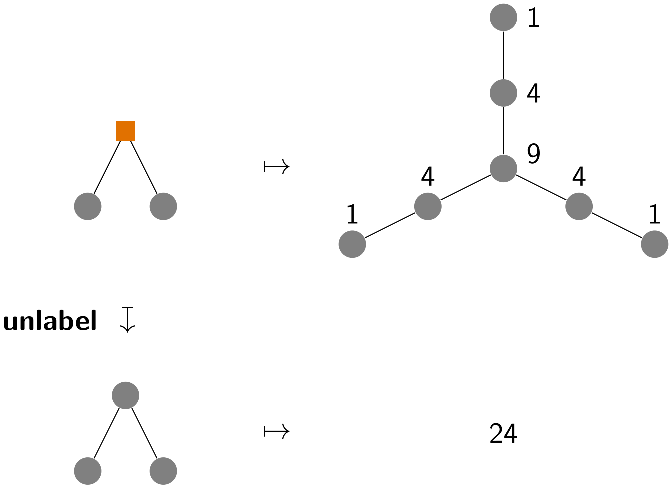

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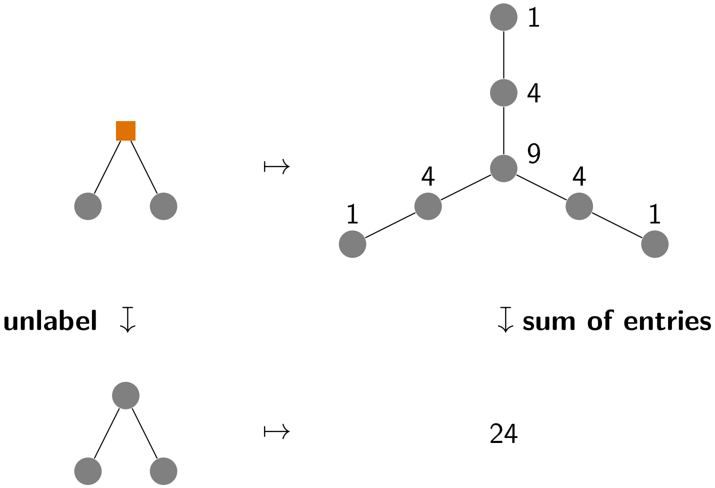
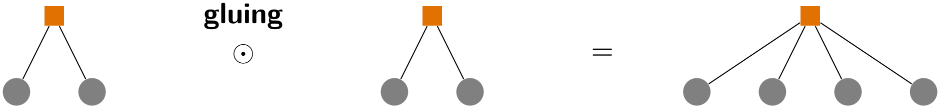


Combinatorial and Algebraic Operations: Gluing and Schur Product



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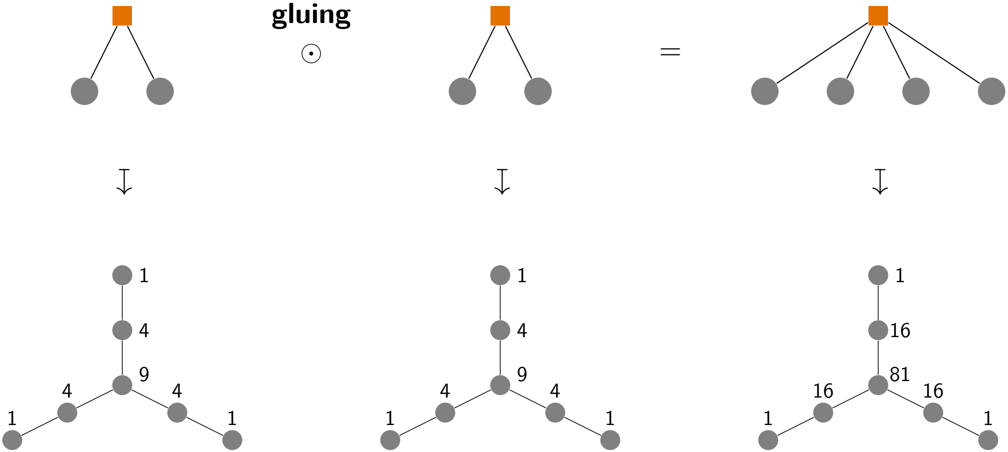

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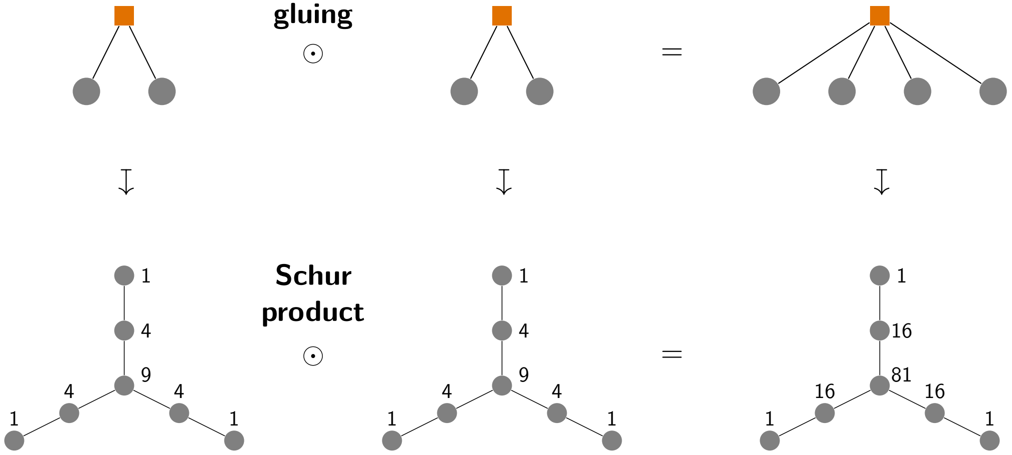

Example: Trees

1. Construct family $\mathcal{F}$ of labelled graphs
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1. Construct family $\mathcal{F}$ of labelled graphs
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4. Define representation and recover system of equations
   - homomorphism tensor representation
   - some linear algebra and representation theory developed in Grohe et al. (2022)
   - Fractional Isomorphism

Glue(ten) intolerance

Can the recipe be used to cook up systems of equations for other graph classes in a generic fashion?

The ingredients labelling, operations, finite generation, and representation have to blend together well.

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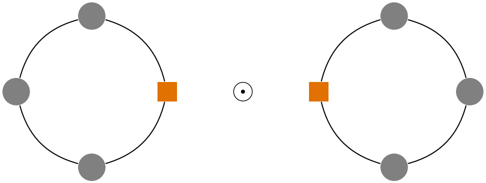

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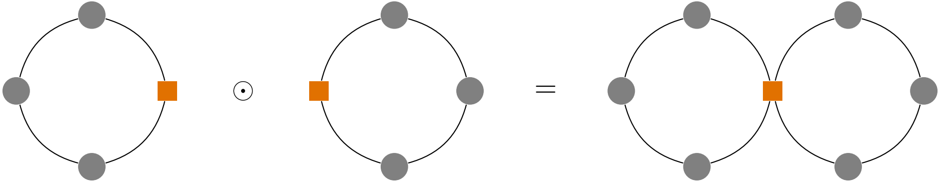

Modern Cuisine: New Recipes

Theorem (Abramsky et al. (2022))

For every "natural" graph class $\mathcal{F}$ there exists a comonad $\mathfrak{C}$ on Graph such that

$$F \in \mathcal{F} \text{ if and only if } F \text{ has a } \mathfrak{C}\text{-coalgebra.}$$

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   - homomorphism tensor representation $\otimes$ label representation
   - some linear algebra and representation theory...

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   - **requires instance-specific arguments**
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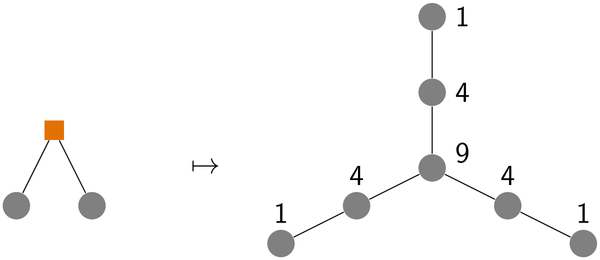
Label Representation

$$\mathcal{F} \longrightarrow \mathbb{C}^{V(G)}$$



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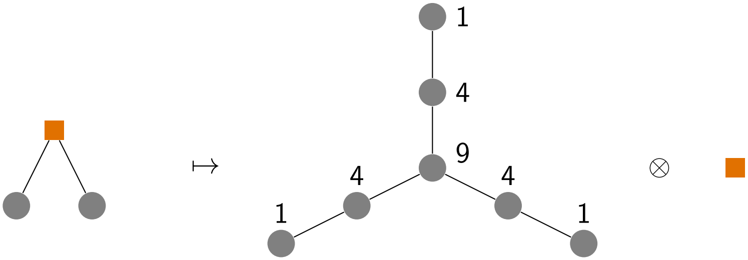

Conclusion

- Comonads on Graph provide new recipes for systems of equations capturing homomorphism indistinguishability
  - labelling, operations, and representations are governed by comonad
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- Comonads on Graph provide new recipes for systems of equations capturing homomorphism indistinguishability
  - labelling, operations, and representations are governed by comonad
  - finite generation yet requires instance-specific arguments
- This yields novel systems of equations characterising $\mathsf{C}_k \cap \mathsf{C}^q$ equivalence
  - check out Rattan and Seppelt (2022) `arXiv:2103.02972`!

Bibliography I

Abramsky, S., Jakl, T., and Paine, T. (2022). Discrete density comonads and graph parameters. Number: arXiv:2205.06589 arXiv:2205.06589 [cs, math].

Dawar, A., Jakl, T., and Reggio, L. (2021). Lovász-type theorems and game comonads. In *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*, pages 1–13. IEEE.

Grohe, M., Rattan, G., and Seppelt, T. (2022). Homomorphism Tensors and Linear Equations. In Bojańczyk, M., Merelli, E., and Woodruff, D. P., editors, *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*, volume 229 of *LIPIcs*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Mančinska, L. and Roberson, D. E. (2020). Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 661–672. IEEE.

Montacute, Y. and Shah, N. (2021). The Pebble-Relation Comonad in Finite Model Theory. *arXiv:2110.08196 [cs, math]*. arXiv: 2110.08196.

Bibliography II

Rattan, G. and Seppelt, T. (2022). Weisfeiler–Leman and Graph Spectra. Number: arXiv:2103.02972 arXiv:2103.02972 [cs, math].

Reggio, L. (2021). Polyadic Sets and Homomorphism Counting. *arXiv:2110.11061 [cs, math]*. arXiv: 2110.11061.

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