How to Compose Shortest Paths

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Problem Statement

Suppose that G and H are weighted graphs

$$G : X \times X \to [0, \infty]$$

$$H : Y \times Y \to [0, \infty]$$

sharing a subset of vertices

$$\begin{tikzcd} X & & Y \\ & K \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Pushforwards and Discrete Graphs

Given a function $f\colon X \to Y$ we may pushforward G along f to get

$$f_*(G)\colon Y \times Y \to [0, \infty]$$

$$f_*(G)(y, y') = \min_{(x,x') \in f^{-1}(y,y')} G(x, x')$$

For a set X, there is a weighted graph

$$LX\colon X \times X \to [0, \infty]$$

given by

$$LX(i, j) = \infty$$

for all i and j.

$$
\begin{tikzcd}
 & X +_K Y & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "j"'] \\
 & K \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G +_{L(K)} Y & \\
G \arrow[ur] & & H \arrow[ul] \\
 & L(K) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Proposition: In the category $[0, \infty]$Graph the pushout is given by

$$G +_{LK} H(x, y) = \min\{i_*(G)(x, y), j_*(H)(x, y)\}$$

$$\begin{tikzcd}
 & X +_K Y & \\
X \arrow[ur, "i"] & & Y \arrow[ul, "j"'] \\
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Proposition: In the category $[0, \infty]$Graph the pushout is given by

$$G +_{LK} H(x, y) = \min\{i_*(G)(x, y), j_*(H)(x, y)\}$$

Moral: To compose two graphs, we first extend them to the shared domain and then take pointwise minimum.

Shortest Paths

G is thought of as a matrix with the $G(i,j)$ as entries. These graphs may be multiplied as

$$G \cdot H(i,j) = \min_{k \in X}\{G(i,k) + H(k,j)\}$$

or added as

$$G + H(i,j) = \min\{G(i,j), H(i,j)\}$$

Proposition: All pairs shortest paths in G are found as

$$F(G) = \sum_{n \geq 0} G^n$$

with operations as above.

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Intuition: G^n records the shortest paths of length n.

Problem Statement!

The composition problem for F asks: given $F(G)$ and $F(H)$ as inputs, find $F(G +_{LK} H)$.

A Solution

Theorem:

$$F(G +_{L(K)} H) =$$

$$\sum_{n \leq |K|} \underbrace{\mathsf{F}(\mathsf{G})\mathsf{F}(\mathsf{H})\mathsf{F}(\mathsf{G})\dots}_{n \text{ times}} + \underbrace{\mathsf{F}(\mathsf{H})\mathsf{F}(\mathsf{G})\mathsf{F}(\mathsf{H})\dots}_{n \text{ times}}$$

where $\mathsf{F}(\mathsf{G})$ and $\mathsf{F}(\mathsf{H})$ are the pushforwards $i_*(F(G))$ and $j_*(F(H))$ to the domain $X +_K Y$.

Theorem:

$$F(G +_{L(K)} H) =$$

$$\sum_{n \leq |K|} \underbrace{\mathsf{F(G)F(H)F(G)}\ldots}_{n \text{ times}} + \underbrace{\mathsf{F(H)F(G)F(H)}\ldots}_{n \text{ times}}$$

where F(G) and F(H) are the pushforwards $i_*(F(G))$ and $j_*(F(H))$ to the domain $X +_K Y$.

Proof Idea: Each term of the sum accounts for zig-zags of length n.

Composition Symbols

Let

$$\mathsf{F}(\mathsf{G}) = \begin{bmatrix} GG & GK & 0 \\ KG & KK_G & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathsf{F}(\mathsf{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & KK_H & KH \\ 0 & HK & HH \end{bmatrix}$$

Plugging these matrices into the theorem when $|K| = k$ give the composition symbols, Symbol(i, j, k), as entries of the result.

$$\text{Symbol}(4, 1, 3) = GK \cdot KH + GK \cdot KK_H \cdot KK_G \cdot KH$$

Composition Symbols

Let

$$F(G) = \begin{bmatrix} GG & GK & 0 \\ KG & KK_G & 0 \\ 0 & 0 & 0 \end{bmatrix} \; F(H) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & KK_H & KH \\ 0 & HK & HH \end{bmatrix}$$

Plugging these matrices into the theorem when $|K| = k$ give the composition symbols, $\text{Symbol}(i, j, k)$, as entries of the result.

$$\text{Symbol}(4, 1, 3) = GK \cdot KH + GK \cdot KK_H \cdot KK_G \cdot KH$$

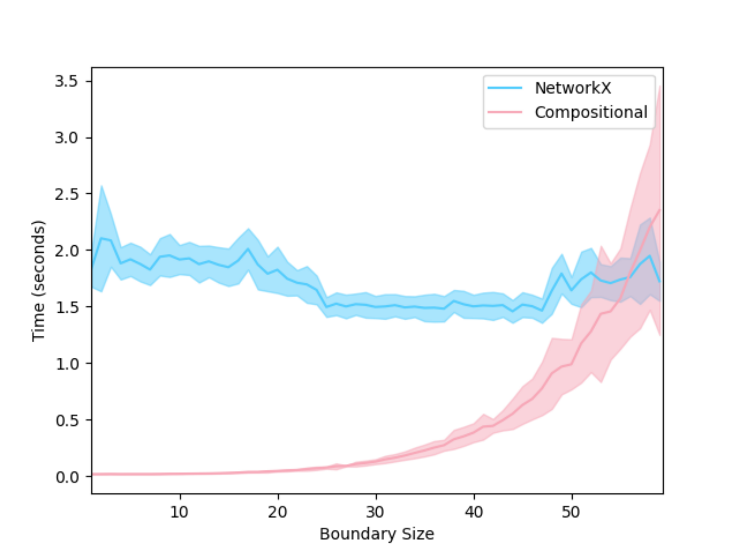
And these symbols keep track of the possible zig-zags between components.

An Algorithm

Suppose that a and b are vertices of $G +_{L(K)} H$ and suppose that we have already found $F(G)$ and $F(H)$. To find the shortest path from a to b:

1. pushoforward and break up $\mathsf{F(G)}$ and $\mathsf{F(H)}$ into blocks as shown above.
2. Determine whether a and b live in G, H, or K and look up the appropriate composition symbol $\mathrm{Symbol}(i, j, k)$.
3. Plug the blocks from before into this composition symbol with a row vector for the first term and a column vector for the last term. The result is the shortest path from a to b.

Results



NetworkX
Compositional
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Time (seconds)
10
20
30
40
50
Boundary Size

Speed-up is most dramatic for small boundary and large component graphs.
2000 nodes each, boundary size: 5, 50 samples, compositional algorithm: 0.1602 ± 0.0169. Djikstra's algorithm: 39.7804 ± 3.3561, compositional Algorithm Precompilation: 93.0590.

Conclusion

This should generalize! F is actually parameterized by semirings. For each semiring Q which is a quantale, there is an adjunction

$$\begin{tikzcd} \mathrm{QGraph} \arrow[r, bend left, "F_Q"] \arrow[r, phantom, "\perp"] & \mathrm{QCat} \arrow[l, bend left, "U_Q"] \end{tikzcd}$$

whose left adjoint is given by

$$F_{\mathrm{Q}}(G) = \sum_{n \geq 0} G^n$$

with matrix operations valued in Q.

$F_Q(G)$ finds the solution to the algebraic path problem on G.

poset	plus	multiplication	solution of path problem
$([0,\infty],\geq)$	inf	$+$	shortest paths in a weighted graph
$([0,\infty],\leq)$	sup	inf	maximum capacity in the tunnel problem
$([0,1],\leq)$	sup	$\times$	most likely paths in a Markov process
$\{T,F\}$	OR	AND	transitive closure of a directed graph
$(\mathcal{P}(\Sigma^*),\subseteq)$	$\bigcup$	concatenation	decidable language of a NFA

I hope to generalize the compositional algorithm to all of these problems as well.

Thank you for listening!