From profinite words to profinite λ-terms

Vincent Moreau, joint work with Paul-André Melliès and Sam van Gool

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Context of the talk

Two different kinds of automata:

- Deterministic automata (in **FinSet**)
- Non-deterministic automata (in **FinRel**)

Profinite methods are well established for words using finite monoids.

Contribution: definition of profinite λ-terms in any model and proof that

Profinite words are in bijection with deterministic profinite λ-terms

using the Church encoding of words and Reynolds parametricity.

This leads to a notion of non-deterministic profinite λ-term in **FinRel**.

Interpreting words as λ-terms

Simply typed λ-terms

λ-terms are defined by the grammar

$$M, N \quad ::= \quad x \mid \lambda x.M \mid MN.$$

Simple types are generated by the grammar

$$A, B \quad ::= \quad \mathbb{o} \mid A \Rightarrow B.$$

For simple types, typing derivations are generated by the following three rules:

$$\frac{}{\Gamma, x : A \vdash x : A}\ \texttt{Var} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ \texttt{App} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}\ \texttt{Abs}$$

The Church encoding for words

Any natural number n can be encoded in the simply typed λ-calculus as

$$s : o \Rightarrow o,\ z : o \vdash \underbrace{s\,(\ldots(s\ z))}_{n \text{ applications}} : o.$$

A natural number is just a word over a one-letter alphabet.

For example, the word *abba* over the two-letter alphabet $\{a, b\}$

$$a : o \Rightarrow o,\ b : o \Rightarrow o,\ c : o \vdash a(b(b(ac))) : o.$$

is encoded as the closed λ-term

$$\lambda a.\lambda b.\lambda c.a(b(b(ac)))\ :\ \underbrace{(o \Rightarrow o)}_{\text{letter } a} \Rightarrow \underbrace{(o \Rightarrow o)}_{\text{letter } b} \Rightarrow \underbrace{o}_{\text{input}} \Rightarrow \underbrace{o}_{\text{output}}.$$

Categorical interpretation

Let **C** be a cartesian closed category.

In order to interpret the simply typed λ-calculus in **C**, we pick an object Q of **C** in order to interpret the base type $\mathtt{o}$ and define, for any simple type A, the object

$$[\![A]\!]_Q$$

by induction, as follows:

$$\begin{array}{rcl} [\![\mathtt{o}]\!]_Q & := & Q \\ [\![A \Rightarrow B]\!]_Q & := & [\![A]\!]_Q \Rightarrow [\![B]\!]_Q. \end{array}$$

The simply typed λ-terms are then interpreted by structural induction on their type derivation using the cartesian closed structure of **C**.

The category FinSet

Fact. The category **FinSet** is cartesian closed: there is a bijection natural in A and C

$$\mathbf{FinSet}(A \times B, C) \quad \cong \quad \mathbf{FinSet}(B, A \Rightarrow C)$$

where $A \Rightarrow C$ is the set of functions from A to C.

In particular, given a finite set Q used to interpret $\mathtt{o}$, every word w over the alphabet $\Sigma = \{a, b\}$ seen as a λ-term

$$\vdash w \;:\; \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{\text{letter } a} \Rightarrow \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{\text{letter } b} \Rightarrow \underbrace{\mathtt{o}}_{\text{input}} \Rightarrow \underbrace{\mathtt{o}}_{\text{output}}$$

can be interpreted in **FinSet** as

$$[\![w]\!]_Q \quad \in \quad (Q \Rightarrow Q) \Rightarrow (Q \Rightarrow Q) \Rightarrow Q \Rightarrow Q$$

which describes how the word will interact with a deterministic automaton.

Entering the profinite world

Profinite words

Definition. A **profinite word** is a family of maps

$$u_M \quad : \quad \mathrm{Hom}(\Sigma^*, M) \longrightarrow M \qquad \text{where } M \text{ ranges over all finite monoids}$$

such that for every pair of homomorphisms $p : \Sigma^* \to M$ and $f : M \to N$, with M and N finite monoids, we have $u_N(f \circ p) = f(u_M(p))$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(\Sigma^*, M) & \xrightarrow{f \circ -} & \mathrm{Hom}(\Sigma^*, N) \\ {\scriptstyle u_M}\downarrow & & \downarrow{\scriptstyle u_N} \\ M & \xrightarrow[f]{} & N \end{array} .$$

Remark. Any word $w \in \Sigma^*$ induces a profinite word u whose components are

$$u_M \quad : \quad p \longmapsto p(w)$$

for every finite monoid M.

Key property: parametricity of profinite words

Definition. Given M, N two finite monoids and $R \subseteq M \times N$, we say that R is a **monoidal relation** $M \nrightarrow N$ if it is a submonoid of $M \times N$. This means that

$$(e_M, e_N) \in R \qquad \text{and} \qquad \text{for all } (m,n) \text{ and } (m',n') \text{ in } R, \text{ we have } (mm', nn') \in R.$$

Proposition. Let $u = (u_M)$ be a family of maps. The following are equivalent:

- u is profinite
- for every pair of homomorphisms $p : \Sigma^* \to M$ and $q : \Sigma^* \to N$ with M and N finite monoids, and for any monoidal relation $R : M \nrightarrow N$,

$$\text{if for all } w \in \Sigma^* \text{ we have } (p(w), q(w)) \in R, \qquad \text{then } (u_M(p), u_N(q)) \in R.$$

Parametric λ-terms

Definition of logical relations

Recall that for any set Q we have defined the set

$$[\![A]\!]_Q$$

by structural induction on the type A.

We extend the construction to set-theoretic relations $R : P \nrightarrow Q$, giving a relation

$$[\![A]\!]_R \quad : \quad [\![A]\!]_P \nrightarrow [\![A]\!]_Q \ .$$

by structural induction on the type A:

$$\begin{aligned} [\![\mathrm{o}]\!]_R \quad &:= \quad R \\ [\![A \Rightarrow B]\!]_R \quad &:= \quad \{\, (f, g) \in [\![A \Rightarrow B]\!]_P \times [\![A \Rightarrow B]\!]_Q \mid \\ &\qquad \text{for all } x \in [\![A]\!]_P \text{ and } y \in [\![A]\!]_Q \ , \\ &\qquad\qquad \text{if } (x, y) \in [\![A]\!]_R \text{ then } (f(x), g(y)) \in [\![B]\!]_R \,\}. \end{aligned}$$

Definition. Let A be a simple type. A **parametric λ-term** of type A is a family of elements

$$u_Q \quad \in \quad [\![A]\!]_Q \qquad \text{where } Q \text{ ranges over all finite sets,}$$

such that, for every binary relation $R \colon P \nrightarrow Q$, we have

$$(u_P, u_Q) \quad \in \quad [\![A]\!]_R \,.$$

Theorem. Parametric λ-terms define a cartesian closed category, and the parametric λ-terms of type

$$\mathrm{Church}_\Sigma \quad := \quad \underbrace{(\mathrm{o} \Rightarrow \mathrm{o}) \Rightarrow \ldots \Rightarrow (\mathrm{o} \Rightarrow \mathrm{o})}_{|\Sigma| \text{ times}} \Rightarrow (\mathrm{o} \Rightarrow \mathrm{o})$$

are in bijection with the profinite words on Σ.

Conclusion

Current & future work:

- find a syntax for all the parametric λ-terms of any type in the functional model;
- determine the parametric λ-terms of type Church_Σ in the relational model;
- investigate a generalization of logic on words with **MSO** to a logic on λ-terms.

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Thank you for your attention!

Any questions?

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Cartesian closed categories

The λ-calculus is about applying functions to arguments.

The simply typed λ-calculus is interpreted using cartesian closed categories.

A cartesian closed category $\mathbf{C}$ is a category:

- with finite products
- such that for every object A, the functor

$$A \times - : \mathbf{C} \to \mathbf{C}$$

  has a right adjoint

$$A \Rightarrow - : \mathbf{C} \to \mathbf{C}.$$

This is the categorified version of an implicative $\wedge$-semilattice.

Proof that profinite words are parametric

Para $\implies$ Pro. Let $p : \Sigma^* \to M$. Any morphism $u : M \to N$ induces a monoidal relation $R : M \nrightarrow N$ which is its graph. By parametricity, $u_N(f \circ p) = f(u_M(p))$.

Pro $\implies$ Para. Let $p : \Sigma^* \to M$ and $q : \Sigma^* \to N$ be homomorphisms and $R : M \nrightarrow N$ be a monoidal relation such that

$$\text{for all } w \in \Sigma^*, \qquad (p(w), q(w)) \in R.$$

The monoidal relation R induces a submonoid $i : S \hookrightarrow M \times N$. Because of the above-stated property, there is $h : \Sigma^* \to S$ such that $i \circ h = \langle p, q \rangle$. Therefore,

$$\begin{aligned}(u_M(p), u_N(q)) &= (\pi_1(u_{M\times N}(\langle p, q\rangle)), \pi_2(u_{M\times N}(\langle p, q\rangle))) \\ &= u_{M\times N}(\langle p, q\rangle) \\ &= i(u_S(h)).\end{aligned}$$

We obtain that $(u_M(p), u_N(q)) \in R$, so u is parametric. $\square$

The inverse bijections T and W

Pro $\to$ Para. Every profinite word u induces a parafinite term with components

$$T(u)_Q \quad : \quad \begin{array}{ccc} \Sigma \Rightarrow (Q \Rightarrow Q) & \longrightarrow & Q \Rightarrow Q \\ p & \longmapsto & u_{Q \Rightarrow Q}(p) \end{array}$$

given the fact that $Q \Rightarrow Q$ is a monoid for the function composition.

Para $\to$ Pro. Every parametric term θ induces a profinite word with components

$$W(\theta)_M \quad : \quad \begin{array}{ccc} \Sigma \Rightarrow M & \longrightarrow & M \\ p & \longmapsto & \theta_M(i_M \circ p)(e_M) \end{array}$$

$$\begin{array}{ccc} \Sigma \Rightarrow (M \Rightarrow M) & \xrightarrow{\theta_M} & M \Rightarrow M \\ {\scriptstyle i_M \circ -}\uparrow & & \downarrow{\scriptstyle -(e_M)} \\ \Sigma \Rightarrow M & \xrightarrow[W(\theta)_M]{- - - - -} & M \end{array}$$

where $i_M : M \to (M \Rightarrow M)$ is the Cayley embedding.

These are bijections between profinite words and parametric λ-terms.

Let u be a profinite word. Recall that $u_M \colon (\Sigma \Rightarrow M) \to M$.

Its associated parametric λ-term $T(u)$ has components

$$T(u)_Q = u_{(Q \Rightarrow Q)}$$

Its associated profinite word $W(T(u))$, for $p : \Sigma \to M$, is equal to

$$W(T(u))_M(p) = T(u)_M(i_M \circ p)(e_M) = u_{(M \Rightarrow M)}(i_M \circ p)(e_M)$$

In order to show that $W(T(u))$ is u, we use the parametricity of profinite words.

We consider the moinoidal logical relation $R \subseteq (M \Rightarrow M) \times M$ defined as

$$R \quad := \quad \{(f, m) \in (M \Rightarrow M) \times M \mid \forall n \in M,\ f(n) = m \cdot n\}$$

Pro → Para → Pro

We have that $(i_M \circ p, p) \in [\![\mathbb{o} \times \cdots \times \mathbb{o}]\!]_R$ because for all $a \in \Sigma$,

$$\text{for all } m \in I, \qquad (i_M \circ p)(a)(m) = p(a) \cdot m.$$

By parametricity of u applied to R, we have that

$$(u_{(M \Rightarrow M)}(i_M \circ p), u_M(p)) \quad \in \quad [\![\mathbb{o} \Rightarrow \mathbb{o}]\!]_R$$

which means, by definition of $[\![\mathbb{o} \Rightarrow \mathbb{o}]\!]_R$, that

$$\text{for all } (f, m) \in R, \quad \text{we have } (u_{(M \Rightarrow M)}(i_M \circ p)(f), u_M(p)(m)) \in R$$

which gives the desired result:

$$W(T(u)) = u_{(M \Rightarrow M)}(i_M \circ p)(e_M) = u_M(p)(m).$$

Let θ be a parafinite term. Recall that $\theta_Q \in (\Sigma \Rightarrow (Q \Rightarrow Q)) \Rightarrow (Q \Rightarrow Q)$.

Its associated profinite word $W(\theta)$ is equal, for $p : \Sigma \to M$, to

$$W(\theta)_M(p) = \theta_M(i_M \circ p)(e_M).$$

Its reassociated parametric λ-term $T(W(\theta))$ has components

$$T(W(\theta))_Q = W_{(Q \Rightarrow Q)}.$$

We want to show that, for all $p : \Sigma \to (Q \Rightarrow Q)$, we have $\theta_Q(p) = T(W(\theta))_Q(p)$, i.e.

$$\text{for all } q_0 \in Q, \qquad \theta_{(Q\Rightarrow Q)}(i_{(Q\Rightarrow Q)} \circ p)(\mathrm{Id}_Q)(q_0) = \theta_Q(p)(q_0)$$

To show that, we introduce, for any $q_0 \in Q$, the logical relation

$$R_{q_0} \quad := \quad \{(f, q) \in (Q \Rightarrow Q) \times Q \mid f(q_0) = q\}.$$

First, we have $(i_{(Q \Rightarrow Q)} \circ p, p) \in [\![(\Phi \Rightarrow \Phi) \times \cdots \times (\Phi \Rightarrow \Phi)]\!]_{R_{q_0}}$ because for all $a \in \Sigma$,

$$\text{for all } (f, q) \in R, \quad \text{we have } (i_{(Q \Rightarrow Q)} \circ p)(a)(f)(q_0) = p(a)(f(q_0)) = p(a)(q)$$

By parametricity of θ, we obtain that $(\theta_{(Q \Rightarrow Q)}(i_{(Q \Rightarrow Q)} \circ p), \theta_Q(p)) \in [\![\Phi \Rightarrow \Phi]\!]_{R_{q_0}}$.
Given the fact that $(\mathrm{Id}_Q, q_0) \in R_{q_0}$ and by definition of $[\![\Phi \Rightarrow \Phi]\!]_{R_{q_0}}$, we obtain that

$$\theta_{(Q \Rightarrow Q)}(i_{(Q \Rightarrow Q)} \circ p)(\mathrm{Id}_Q)(q_0) = \theta_Q(p)(q_0)$$

which concludes the proof. $\square$