DATA STRUCTURES FOR TOPOLOGICALLY SOUND HIGHER-DIMENSIONAL DIAGRAM REWRITING

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Structure Meets Power
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Theoretical Framework:

DIAGRAMMATIC SETS

arXiv: 2007.14505

This work:

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Central Idea of Higher-Dimensional Rewriting

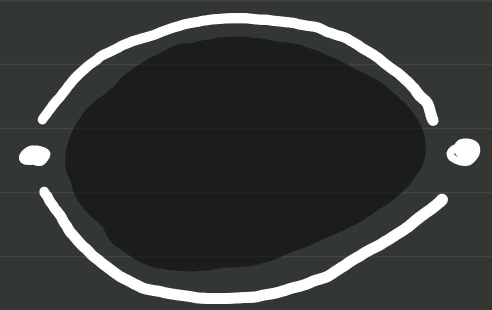
REWRITE SYSTEMS = DIRECTED CELL COMPLEXES

CELL COMPLEX

↳ Space "assembled from n-balls, glued by their (n-1)-sphere boundaries"

Minimal REGULAR cell structure on n-balls:

1. n-dimensional cell,
2. k-dimensional cells for $k<n$



DIRECTED CELL COMPLEX

↳ Space "assembled from DIRECTED n-balls, glued by their $(n-1)$-sphere boundaries"

Minimal REGULAR cell structure on DIRECTED n-balls:

1. n-dimensional cell,
2. k-dimensional cells for $k<n$

A directed n-ball has its boundary subdivided into an INPUT half and an OUTPUT half, each of which is a directed $(n-1)$-ball.

A particular kind of orientation, inspired by higher category theory

$\boxed{1D}$ Abstract Rewrite Systems

$\sim$ Directed Graphs

$\sim$ Directed 1D cell complexes

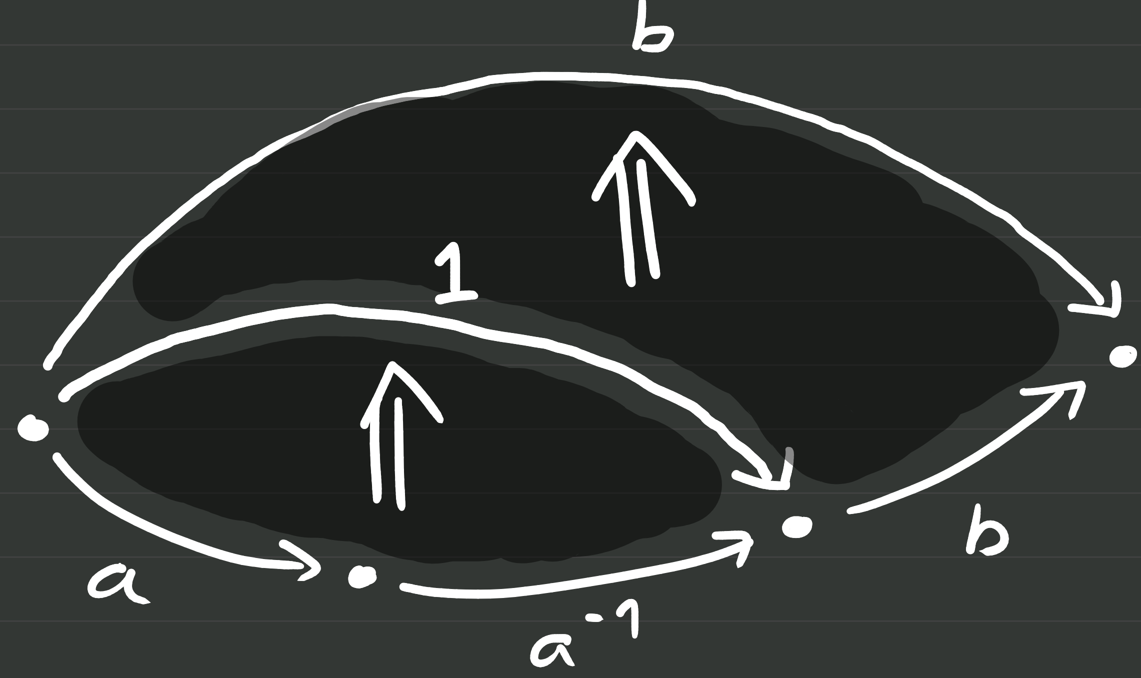
$$
\begin{tikzcd}
 & & z \arrow[dr, bend left] & \\
 & y \arrow[ur, bend left] \arrow[dr, bend right] & & v \\
x \arrow[ur] & & w \arrow[ur] &
\end{tikzcd}
$$

Rewrite sequence $\sim$ Directed homotopy

$\boxed{2D}$ String Rewrite Systems

~ Rewriting paths in graphs

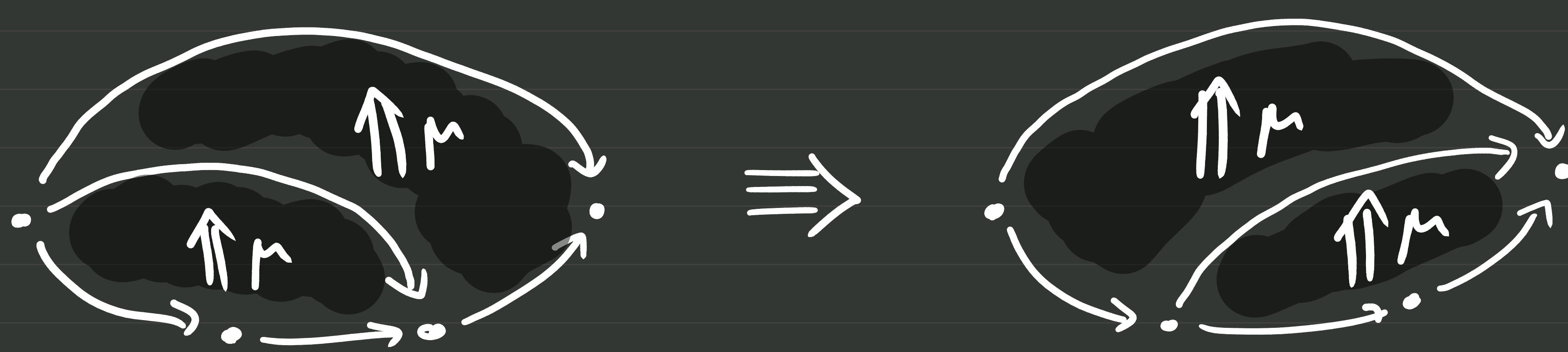
~ Directed 2D cell complexes



$\boxed{3D}$ Term Rewriting Systems

~ Diagram rewriting in PRO(P)s

~ Directed 3D cell complexes



$$\mu(\mu(x,y),z) \Rrightarrow \mu(x,\mu(y,z))$$

Higher Rewrite System $\longrightarrow$ Higher Categories

SYNTAX $\longrightarrow$ SEMANTICS

(HDRSs can be interpreted in higher categories, but they themselves aren't necessarily higher categories)

A VISION:

A model of computation

based on higher-dimensional rewriting

fits naturally with

higher-categorical semantics!

STRUCTURE MEETS POWER

In this work:

1. Data structures for higher rewrite systems & diagrams

2. PTIME algorithm for the isomorphism problem for higher diagrams

Necessary for the naive cost model of a "diagrammatic rewriting machine" to be reasonable

This is all implemented in a Python library, rewalt, also supporting various visualisations

WILL RELEASE SOON!

Check github.com/ahadziha/rewalt in a few weeks

To model a (directed) cell complex,
we need:

1. Models of n-cells & their boundaries

2. Models of "gluing maps", specifying how cells can be put together

TOPOLOGICAL SOUNDNESS:

A directed cell complex also presents a (topological) cell complex.

A well-formed rewrite sequence induces a (cellular) homotopy in the presented space.

DIAGRAMMATIC SETS:

An expressive, yet topologically sound model of HDR.

A classical result of combinatorial topology:

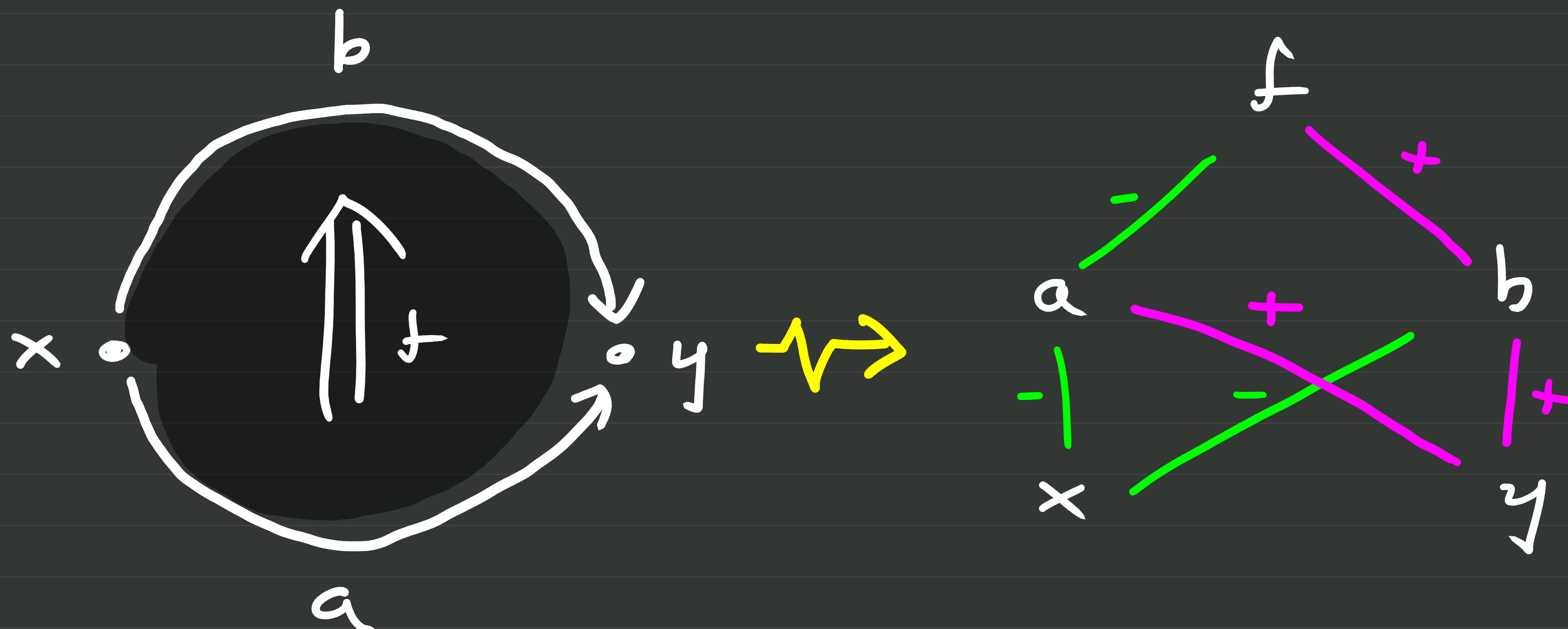
A REGULAR CW COMPLEX is uniquely (up to cellular homeomorphism) described by its FACE POSET

$\downarrow$

FACE POSETS OF REGULAR CW BALLS ARE COMBINATORIAL MODELS OF BALLS

Encode a DIRECTED BALL as its

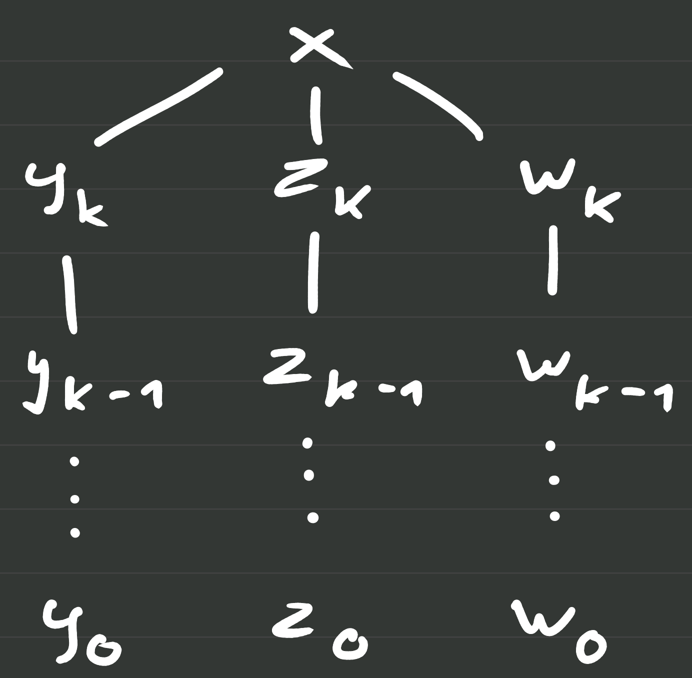
ORIENTED FACE POSET



$-$ Source/Input

$+$ Target/Output

Def A finite poset P is graded if $\forall x \in P$, all maximal descending chains under x have the same length.



Rank / Dimension of x: $k+1$

Def An *oriented graded poset* is a graded poset together with an edge-labelling of its Hasse diagram in $\{-, +\}$.

To encode an o.g. poset is, essentially, to encode its labelled Hasse diagram...

(We use a kind of "graded adjacency list" encoding)

GI := Graph Isomorphism
(not known to be in PTIME)

Proposition

The isomorphism problem for o.g. posets is GI-complete.

The actual "shapes of diagrams" in the model form an inductive subclass of all o.g. posets,

the REGULAR MOLECULES

The class $\mathcal{R}$ of **REGULAR MOLECULES**:

1. **(POINT)** The terminal o.g. poset $\bullet \in \mathcal{R}$.

2. **(ATOM)** If $U, V \in \mathcal{R}$,
   a) $\dim(U) = \dim(V) = n$,
   b) $\partial^{\alpha}_{n-1} U \simeq \partial^{\alpha}_{n-1} V$ for all $\alpha \in \{-, +\}$,
   c) U, V are **ROUND**,
   then $U \Rightarrow V \in \mathcal{R}$.

3. **(PASTE)** If $U, V \in \mathcal{R}$, $\partial^{+}_{k} U \simeq \partial^{-}_{k} V$, then $U \#_k V \in \mathcal{R}$.

Theorem

The isomorphism problem for regular molecules admits a solution running in time $O(n^3 \log n)$.

The solution also gives a "canonical form" / unique representation for regular molecules!

We then formalise higher rewrite systems and diagrams as CONTEXTS and TERMS in a dependent type theory, whose terms are labellings of regular molecules with variables.

("Black-boxing" our implementation of regular molecules)

FUTURE WORK

- Subdiagram search algorithms
- Explicit "diagrammatic machines" & cost models
- rewalt open for contributions & feature requests !

THANKS FOR LISTENING !

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Def Boundaries of a closed subset U

$k \in \mathbb{N}$ $\qquad\qquad$ $\alpha \in \{-, +\}$

$$\Delta_k^\alpha U := \left\{ x \in U \;\middle|\; \dim(x) = k, \text{ and } \forall y \in U \quad \begin{matrix} y \\ | \\ x \end{matrix} \Rightarrow \begin{matrix} y \\ |\alpha \\ x \end{matrix} \right\}$$

$$\mathrm{Max}_j U := \left\{ x \in U \;\middle|\; \dim(x) = j, \text{ and } x \text{ is maximal} \right\}$$

$$\partial_k^\alpha U := \mathrm{cl}\left(\Delta_k^\alpha U\right) \cup \bigcup_{j<k} \mathrm{cl}\left(\mathrm{Max}_j U\right)$$

Notation: for $x \in P$, $\partial_k^\alpha x := \partial_k^\alpha \mathrm{cl}\{x\}$

Def A map $f : P \to Q$ of o.g. posets is a function satisfying

$$\boxed{f(\partial_k^\alpha x) = \partial_k^\alpha f(x)}$$

for all $x \in P$, $k \in \mathbb{N}$, $\alpha \in \{-,+\}$.

Prop A map of o.g. posets is

- order-preserving, and
- dimension-non-increasing.