Presheaves for (P)CSPs

Structure $\wedge$ Power

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Joint work with Samson Abramsky and Anuj Dawar

CSP

(A, B)

$A \to B$

$A \nrightarrow B$

NP-Complete
for every non-unary signature

CSP(B)

A

$A \to B$

$A \nrightarrow B$

Complexity determined by $Pol(B) := \{f \colon B^n \to B\}$
Any non-trivial polymorphism $\implies$ PTIME
Otherwise, NP-Complete
(Bulatov, Zhuk 2017)

PCSP(A,B)

where $A \to B$

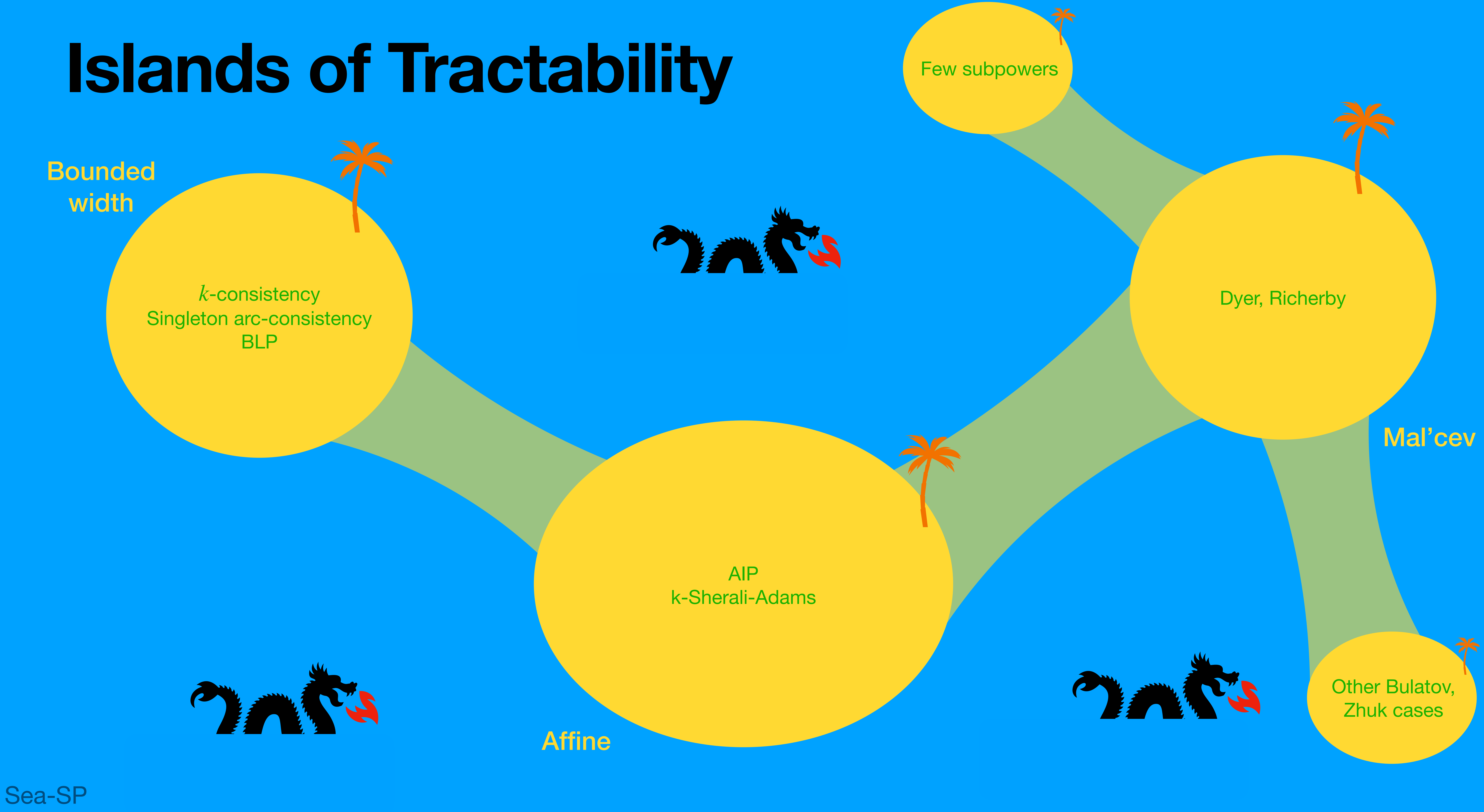
X

$X \to A$

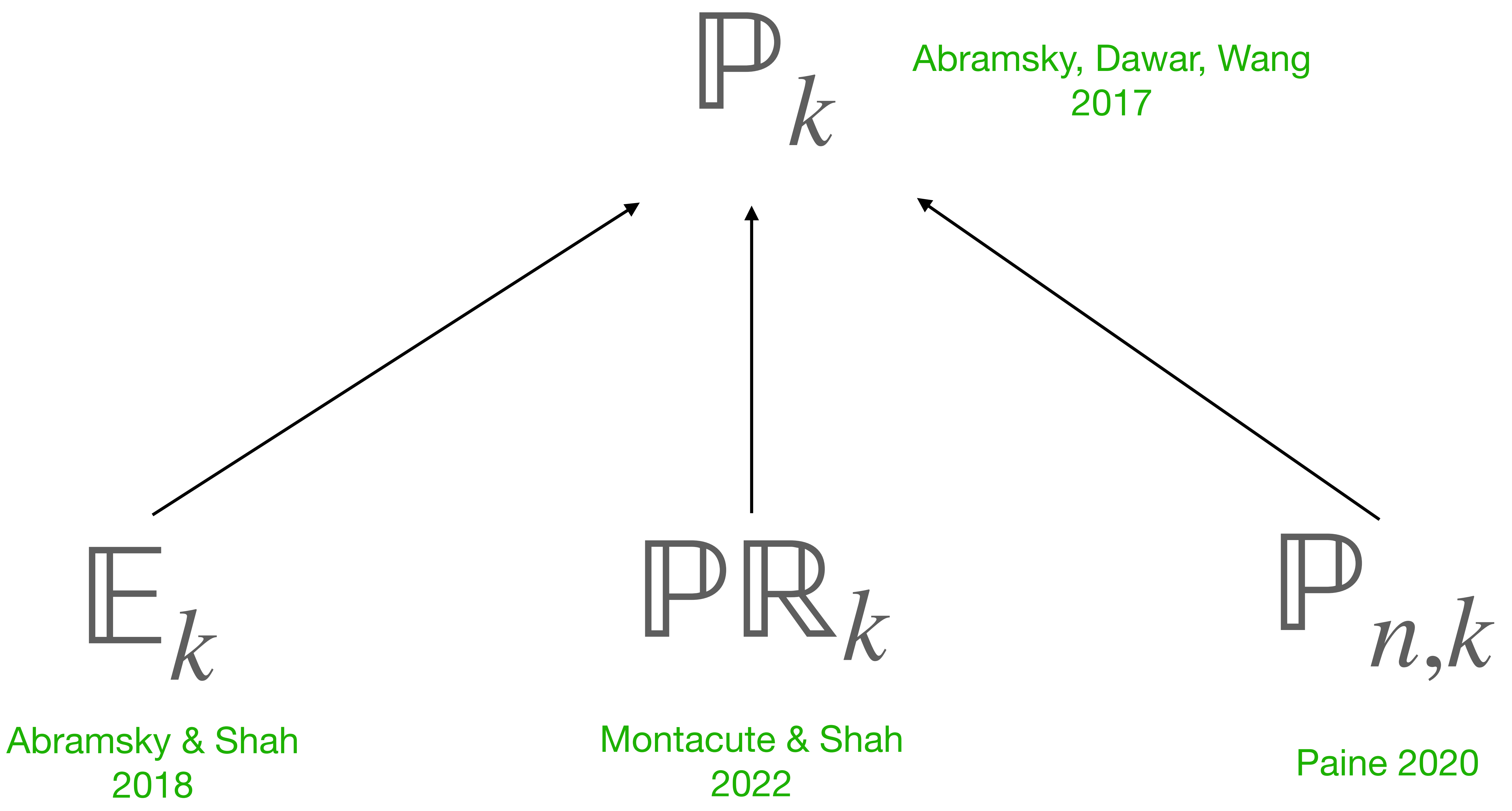
$X \nrightarrow B$

Complexity influenced by $Pol(A, B) := \{f \colon A^n \to B\}$
"Good minion homomorphisms" $\implies$ PTIME
Dichotomy not known, many open problems.
(Brakensiak & Guruswami, Barto, Bulín, Krokhin & Opršal)

Islands of Tractability



Compositional approaches



$\mathbb{P}_k A \to B \iff (A, B)$ is k-consistent

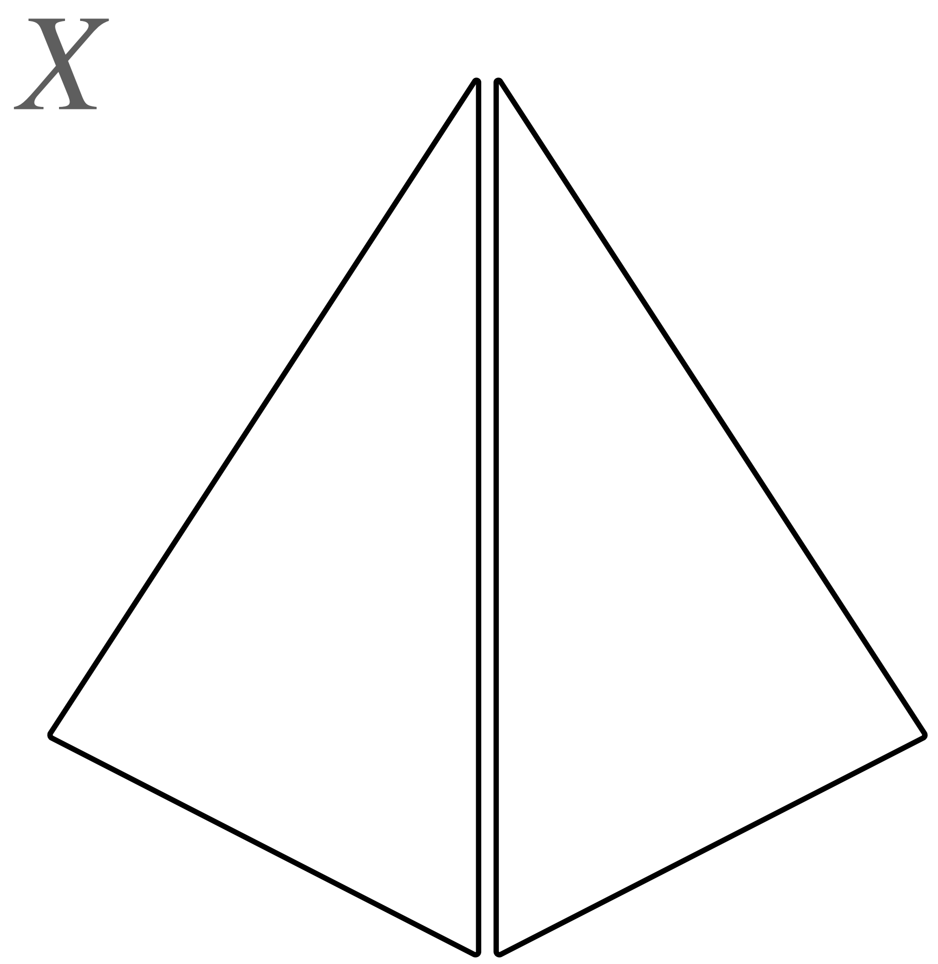
However, all of these lie in the “bounded width” class of CSP algorithms

There have as yet been no comonads which capture other tractable algorithms

Perhaps we need a new approach….

A (quick) introduction to presheaves

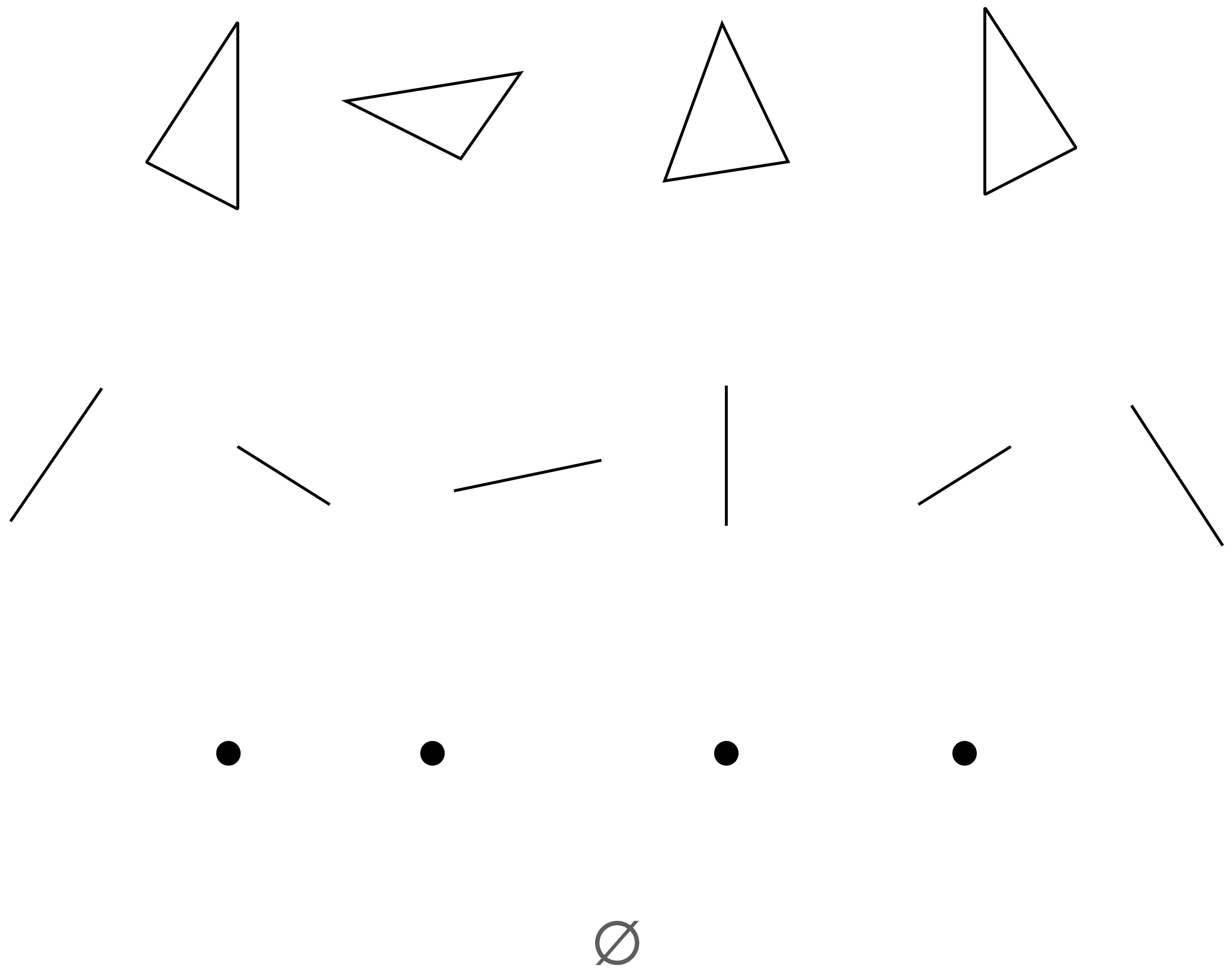
Let X be a space



A (quick) introduction to presheaves

Let X be a space

Let $C \subset \mathscr{P}X$ be some cover of subsets



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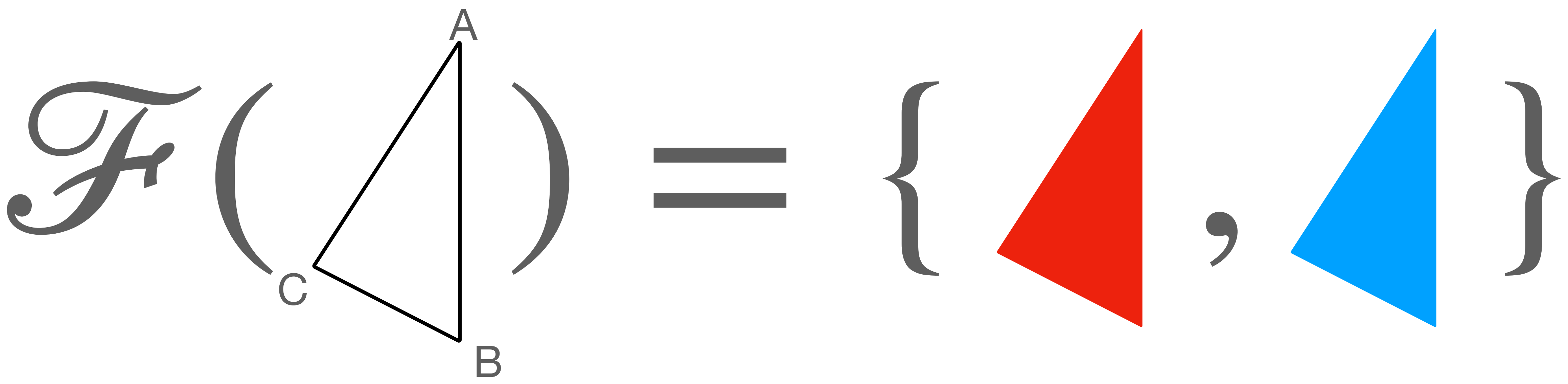

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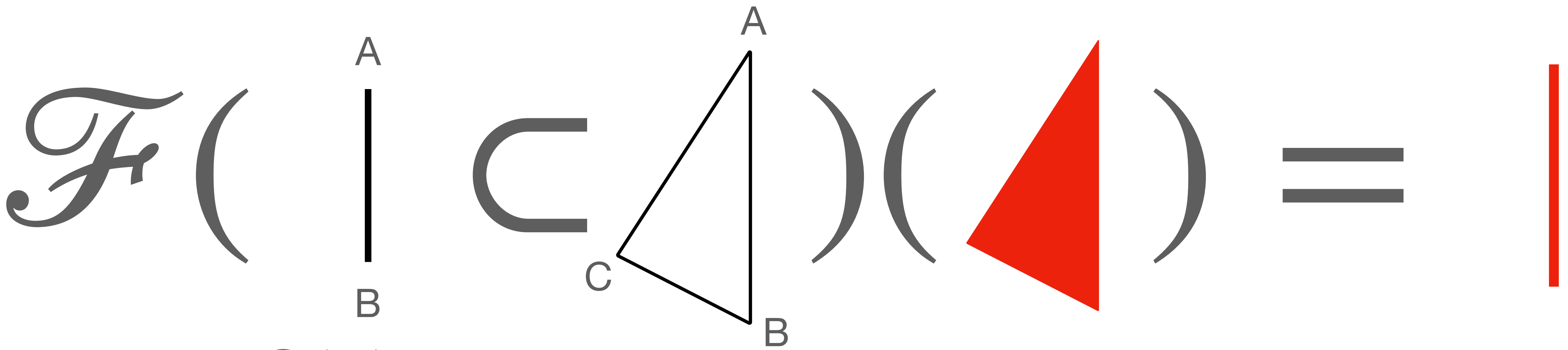

A $\mathbf{Set}$-valued presheaf $\mathscr{F}$ on (X, C) maps:

- “contexts” $U \in C$ to a set of “behaviours” $\mathscr{F}(U)$
- “inclusions” $U' \subset U$ to “restriction” maps $\mathscr{F}(U' \subset U) \colon \mathscr{F}(U) \to \mathscr{F}(U')$

… i.e. $\mathscr{F}$ is a functor $C^{op} \to \mathbf{Set}$

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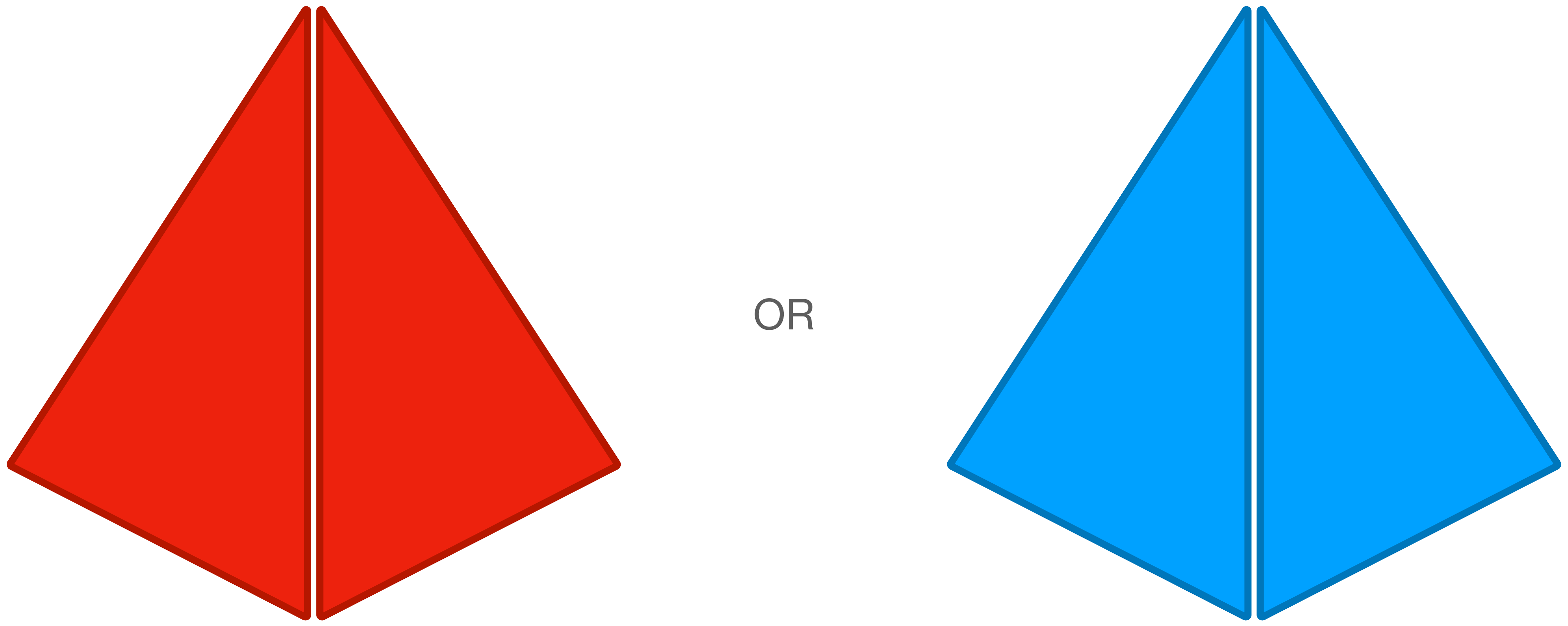

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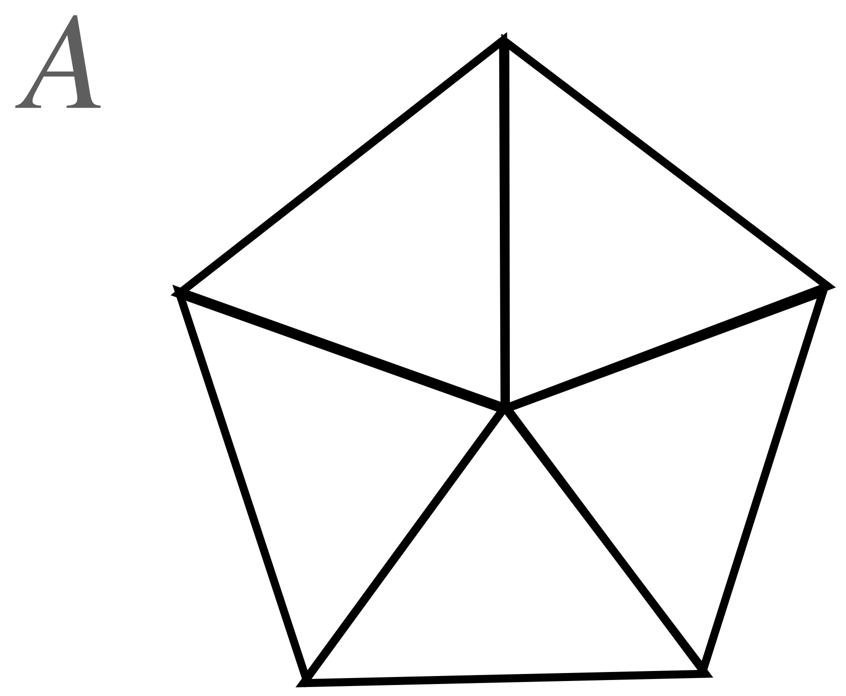
… i.e. $\mathscr{F}$ is a functor $C^{op} \to \mathbf{Set}$

A global section of $\mathscr{F}$ is a natural transformation $s \colon 1 \Rightarrow \mathscr{F}$, so that

- each $s_U \in \mathscr{F}(U)$ is a local section
- when $U' \subset U$ $\;(s_U)|_{U'} = s_{U'}$

Presheaves of local homomorphisms

Let A be a relational structure

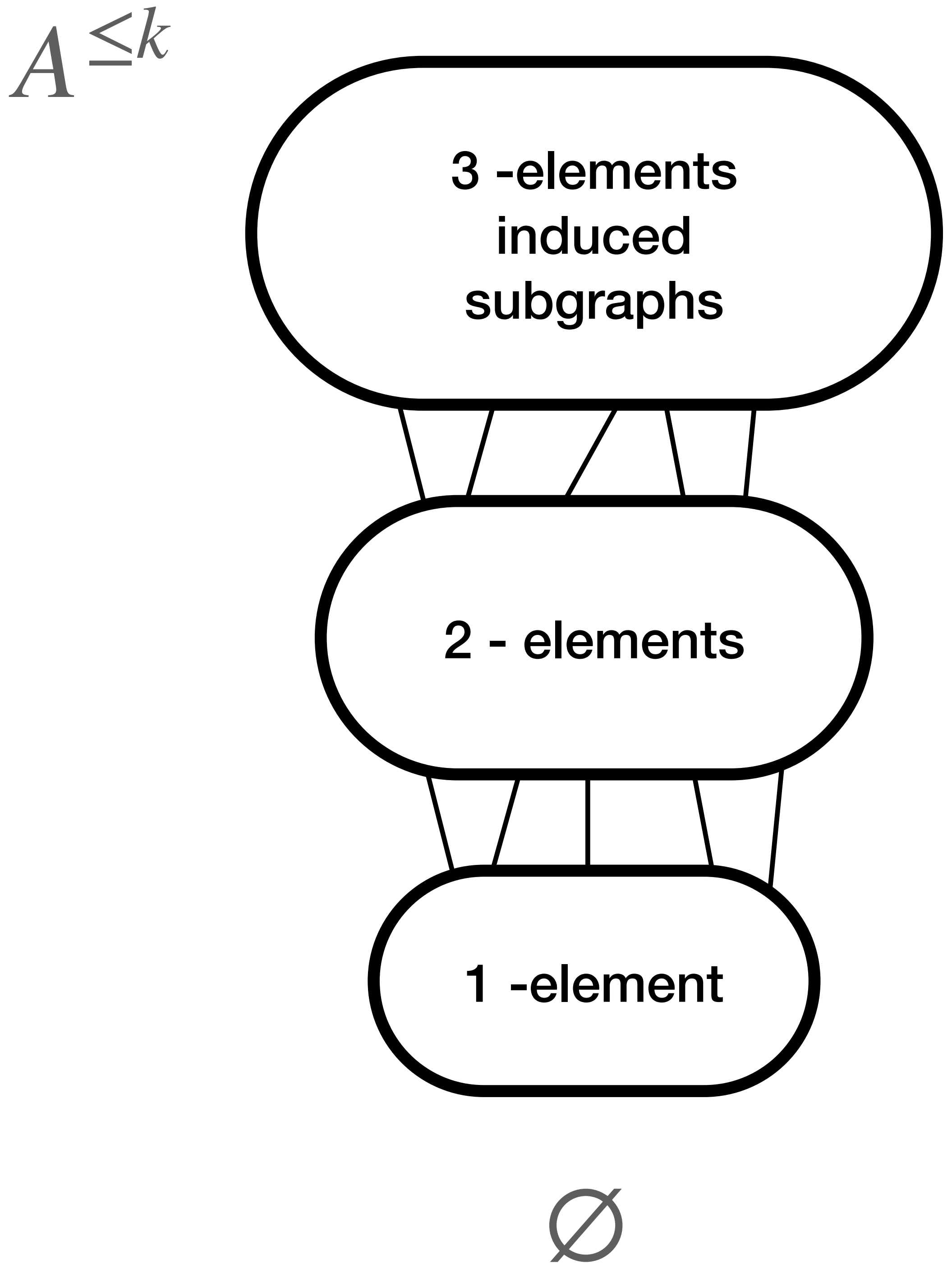


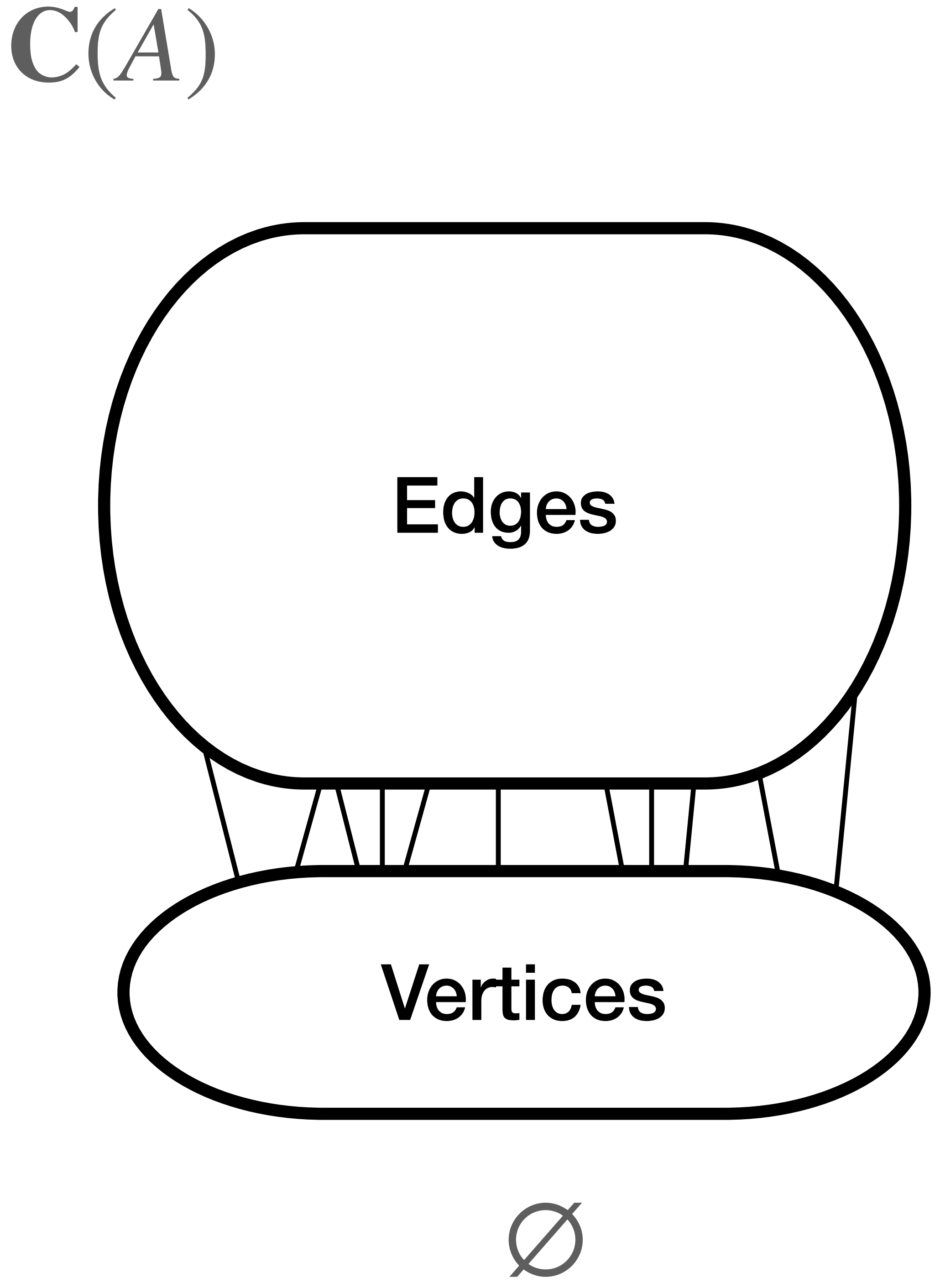
Presheaves of local homomorphisms

Let A be a relational structure

Let C be some cover by substructures of A

- $C = A^{\leq k}$ is the k ***local cover*** of A
- $C = \mathbf{C}(A)$ is the ***arc cover*** of A





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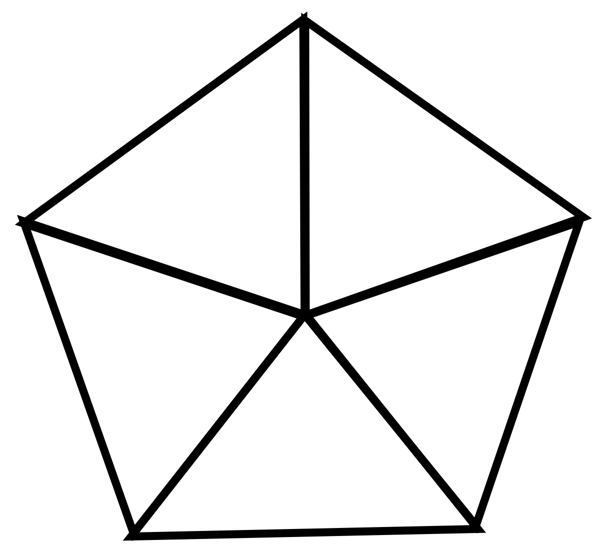
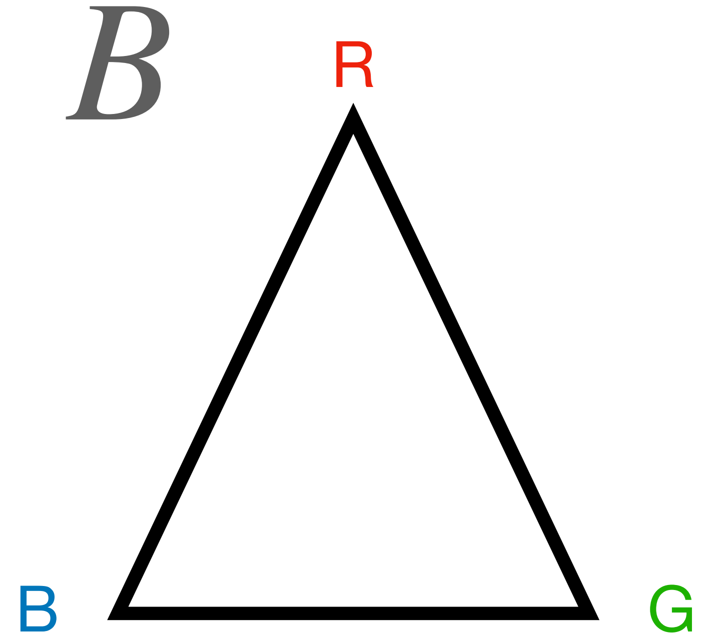


B



$$\mathscr{H}_k(A,B)(U) = \{\text{3-colourings of } U\}$$

$$\mathscr{H}_k(A,B)(U' \subset U)(c) = c_{|_{U'}}$$

For any structure B and either C there is a presheaf which sends U to $\mathbf{Hom}(U,B)$

- $C = A^{\leq k}$ call this $\mathscr{H}_k(A,B)$
- $C = \mathbf{C}(A)$ call this $\mathscr{C}(A,B)$

$\mathscr{H}_k(A, B)$ has a global section

$$\iff$$

There is a homomorphism $A \to B$

This is a difficult (NP-hard) problem.

Abelian presheaves are easier

$\mathscr{H}_k(A, B)$
has a global section

$\Longrightarrow$

$\mathbb{S}\mathscr{H}_k(A, B)$
has a global section

$s \in \mathscr{H}_k(A, B)$ can be extended to a global section

$\Longrightarrow$

$s \in \mathbb{S}\mathscr{H}_k(A, B)$ can be extended to a global section

This is a difficult (NP-hard) problem.

These are tractable for good $\mathbb{S}$

Local-to-global obstructions captured by cohomology

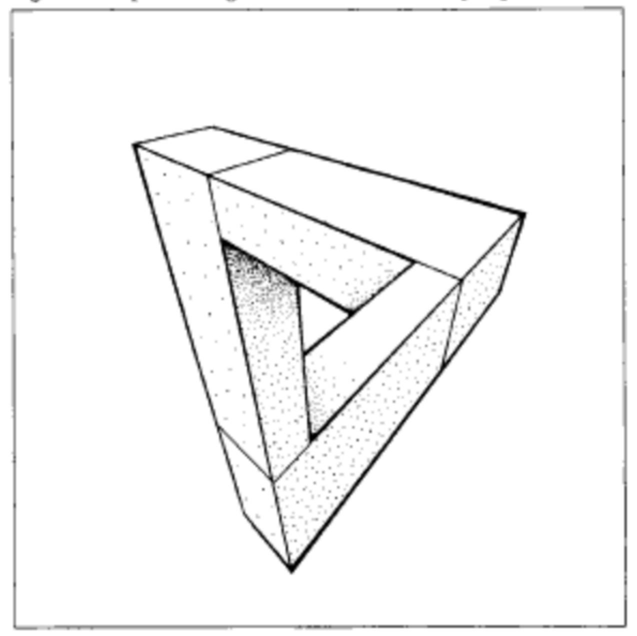
Sur quelques points d'algèbre homologique
Grothendieck (Tohoku 1957)

$$0 \to F_U \to F \to F_Y \to 0$$

$$\cdots H^p_\Phi(X, F_U) \to H^p_\Phi(X, F) \to H^p_{\Phi_Y}(Y, F) \to H^{p+1}_\Phi(X, F_U) \cdots$$

On the cohomology of impossible figures
Penrose (Leonardo 1992)

Fig. 1. An impossible figure, the tribar, drawn in perspective.



Contextuality, Cohomology and Paradox
Abramsky, Barbosa, Kishida, Lal, Mansfield
(CSL 2015)

$\mathcal{S}$ an $\mathbb{S}$-valued presheaf then
$s \in \mathcal{S}$ can be extended to a global section of $\mathcal{S}$ if and only if
a certain cohomological obstruction $\gamma_{\mathbb{S}}(s)$ vanishes.

Local-to-global obstructions captured by cohomology

For each choice of $\mathbb{S}$, 3 algorithms approximating CSP

1."One-Shot"

Does $\mathbb{S}\mathscr{H}_k(A,B)$ have a global section?

2. "Singleton Consistency"

Is there a subpresheaf of $\mathbb{S}\mathscr{H}_k(A,B)$ where $\gamma_{\mathbb{S}}(s)$ vanishes for every singleton assignment ?

3. "Full Cohomological Consistency"

Is there a subpresheaf of $\mathbb{S}\mathscr{H}_k(A,B)$ where $\gamma_{\mathbb{S}}(s)$ vanishes for every s?

Cohomology in Constraint Satisfaction and Structure Isomorphism
AÓC (MFCS 2022)

For any CSP instance (A,B) and $\mathbb{S} = \mathbb{Z}$ these are all PTIME algorithms for fixed k

For $\mathscr{C}(A, B)$

	One-shot	**Singleton Cohomology**	**Full Cohomology**
$\mathbb{B}$	Arc-consistency	SAC	CAC
$\mathbb{Q}^{\geq 0}$	BLP	SBLP	CBLP
$\mathbb{Z}$	AIP	SAIP	CAIP

For $\mathscr{H}_k(A, B)$

	One-shot	**Singleton Cohomology**	**Full Cohomology**
$\mathbb{B}$	k-consistency	?	?
$\mathbb{Q}^{\geq 0}$	k-Sherali-Adams over $\mathbb{Q}^{\geq 0}$	?	?
$\mathbb{Z}$	k-Sherali-Adams over $\mathbb{Z}$	?	Cohomological k-consistency

For $\mathscr{I}_k(A, B)$

	One-shot	Singleton Cohomology	Full Cohomology
$\mathbb{B}$	k-WL	?	?
$\mathbb{Q}^{\geq 0}$	k-Sherali-Adams over $\mathbb{Q}^{\geq 0}$	?	?
$\mathbb{Z}$	k-Sherali-Adams over $\mathbb{Z}$	?	Cohomological k-WL

Advantages of the presheaf approach

- New Algorithms
- Common *compositional* framework for old algorithms
- Connections to other work in CS, maths and beyond
- Exciting new directions for structure and power…

Future work: Power

- How strong are these algorithms?

    1. New ways of proving power upper bounds?
    2. How to compose PCSP algorithms?
    3. A topological approach to Bulatov-Zhuk?

- Relations to logics

    A. Can distinguish properties inexpressible in rank logic.

    B. What is the logic captured by these algorithms?

Future work: Structure

$$A \to B \implies A \to_k^{\mathbb{Z}} B \implies A \to_k B$$

- Structure on the left

    1. Width k tree decompositions as subcovers of $A^{\leq k}$
    2. What structure guarantees $A \to_k^{\mathbb{Z}} B \implies A \to B$?

- Structure on the right

    A. What is the relation with polymorphisms and minions?

    B. Can we characterise "*cohomological width*" of B?

Summary

$A \to B$ $\iff$ A certain presheaf $\mathcal{S}(A, B)$ has a global section $\implies$ $\mathcal{S}(A, B)$ contains subpresheaf with well-behaved $\mathbb{S}$-cohomology

Power

- Strength of algorithms
  1. Upper bounds?
  2. Composition?
  3. Topology x Bulatov-Zhuk?
- Relation to logic
  A. Rank logic?
  B. Logics with new quantifiers?

Structure

- On the left
  1. Decompositions topologically?
  2. Generalised treewidth?
- On the right
  A. Polymorphisms and minions?
  B. "*cohomological width*"?