Monadic
Monadic Second Order Logic

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Structure Meets Power
Paris, 4 July 2022

Structure meets structure

Structure meets structure

Monads

$\eta : \mathrm{Id} \Rightarrow T$

$\mu : TT \Rightarrow T$

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$\mu : TT \Rightarrow T$

Comonads

$\epsilon : D \Rightarrow \mathrm{Id}$

$\delta : D \Rightarrow DD$

Structure meets structure

	Kleisli	Eilenberg-Moore
Monads $\eta : \mathrm{Id} \Rightarrow T$ $\mu : TT \Rightarrow T$		
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Monads $\eta : \mathrm{Id} \Rightarrow T$ $\mu : TT \Rightarrow T$	$X \to TY$ effect to produce	
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Monads $\eta : \mathrm{Id} \Rightarrow T$ $\mu : TT \Rightarrow T$	$X \to TY$ effect to produce	$TX \to X$ structure to compose
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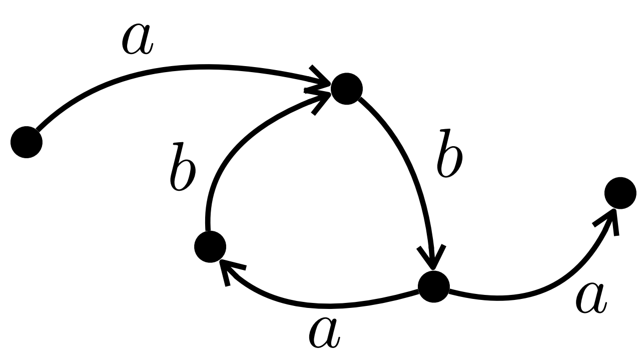
Power meets power

accepted by finite automata



$\Longleftrightarrow$

defined by
regular expressions

$$E ::= \epsilon \mid a \mid E + E \mid EE \mid E^*$$

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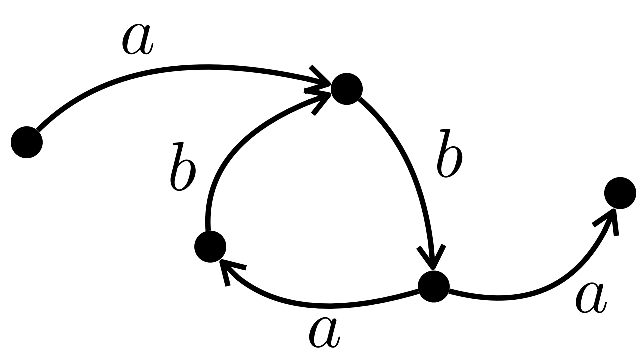

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recognized by finite monoids

$$\overleftarrow{h}(A) = \begin{array}{ccc} L & & A \\ \cap\!| & & \cap\!| \\ \Sigma^* & \xrightarrow[h]{} & M \end{array}$$

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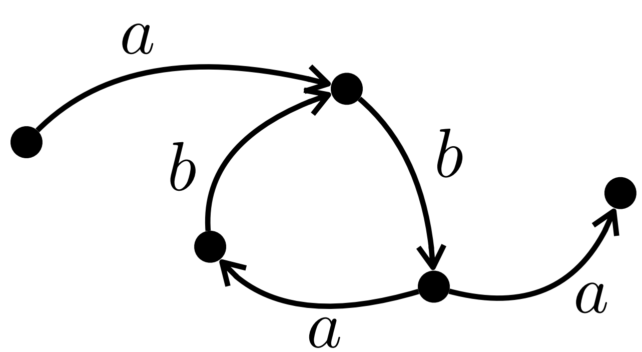

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$\Updownarrow$ $\Updownarrow$

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$$\overleftarrow{h}(A) = L \qquad A$$
$$\quad\;\, \cap\!| \qquad\quad \cap\!|$$
$$\quad\;\, \Sigma^* \xrightarrow[h]{} M$$

$\Longleftrightarrow$

MSO-definable

$$x < y \qquad Q_a(x) \qquad x \in X$$

$$\phi \vee \psi \qquad \neg\phi \qquad \exists X.\phi$$

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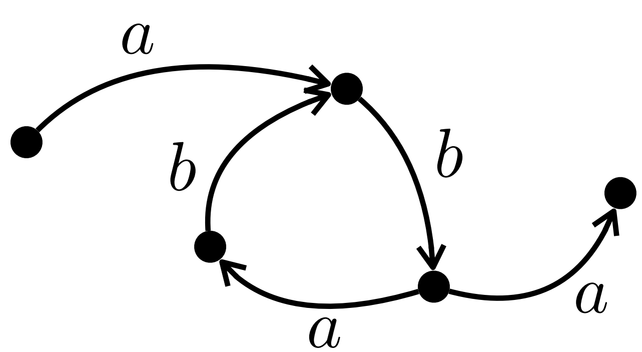

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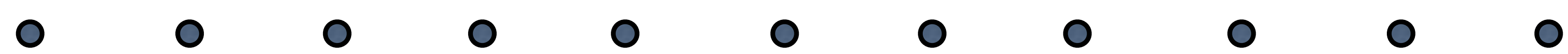
Monadic Second Order Logic

- words as relational structures:

$$a \quad c \quad b \quad c \quad a \quad c \quad b \quad a \quad c \quad a \quad c$$



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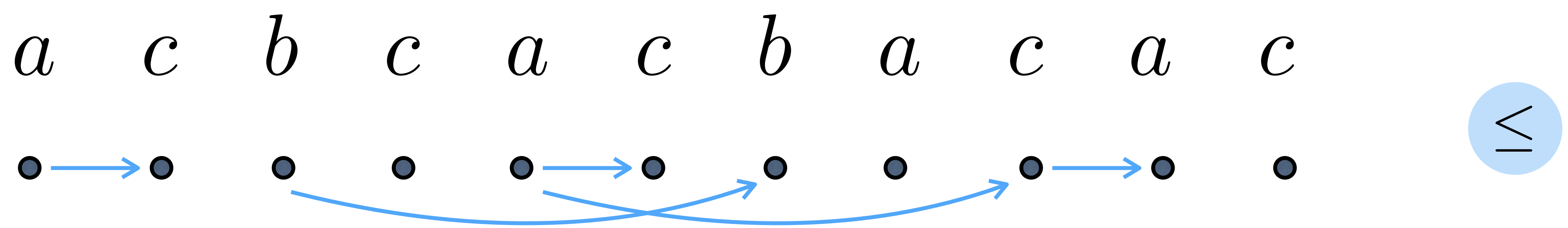

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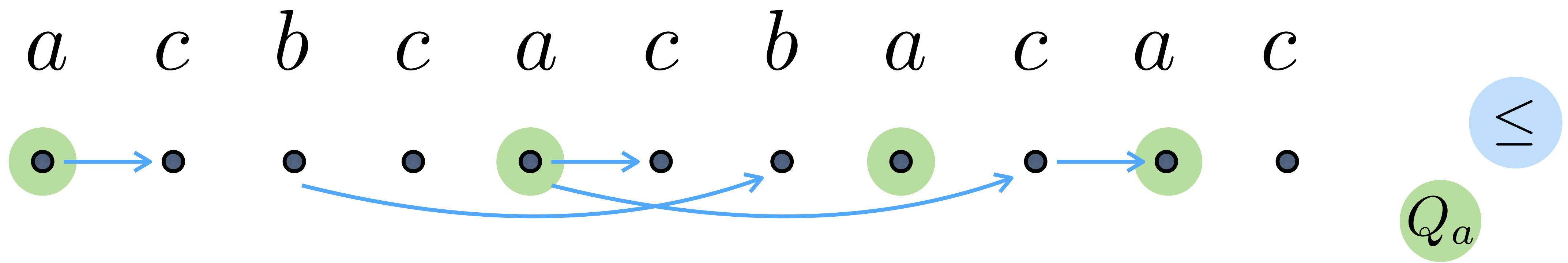

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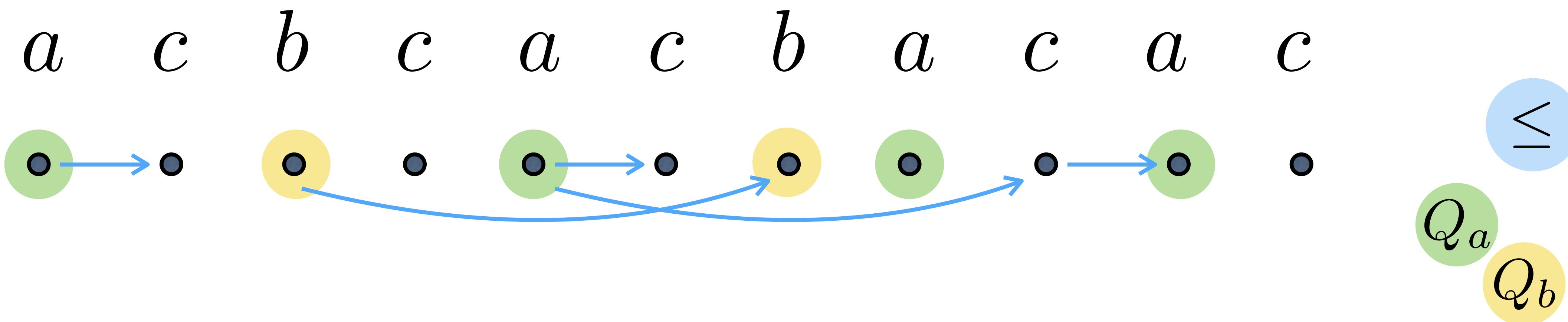

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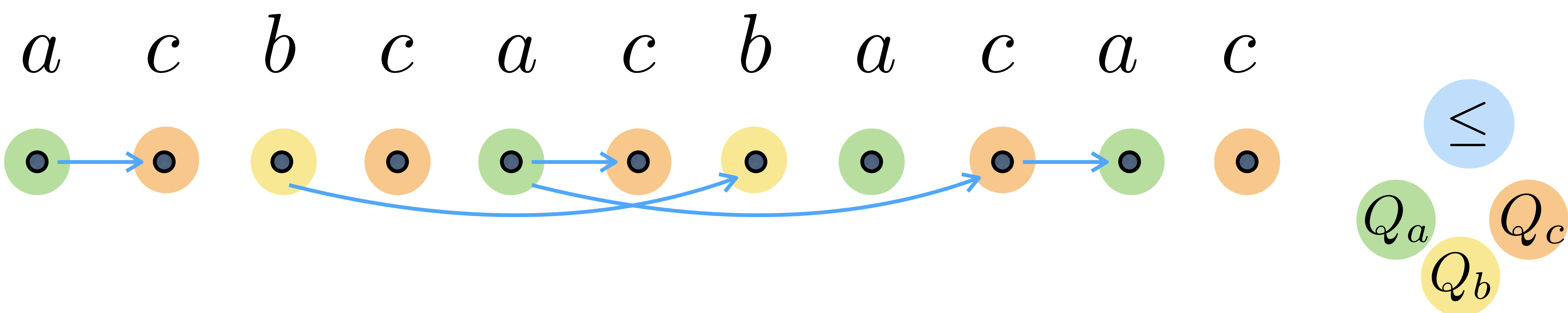

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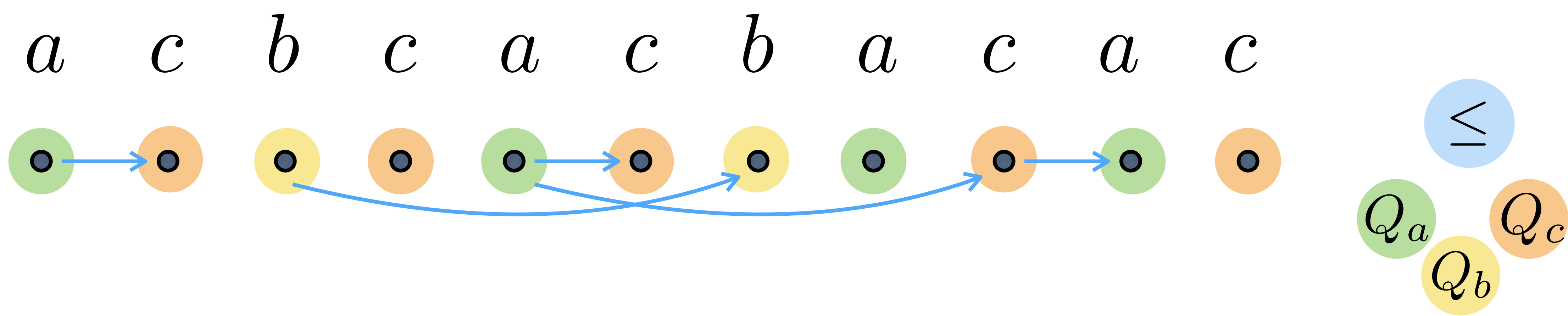

- examples:

$$\forall x. Q_a(x) \Rightarrow \exists y. x < y \wedge Q_c(y)$$

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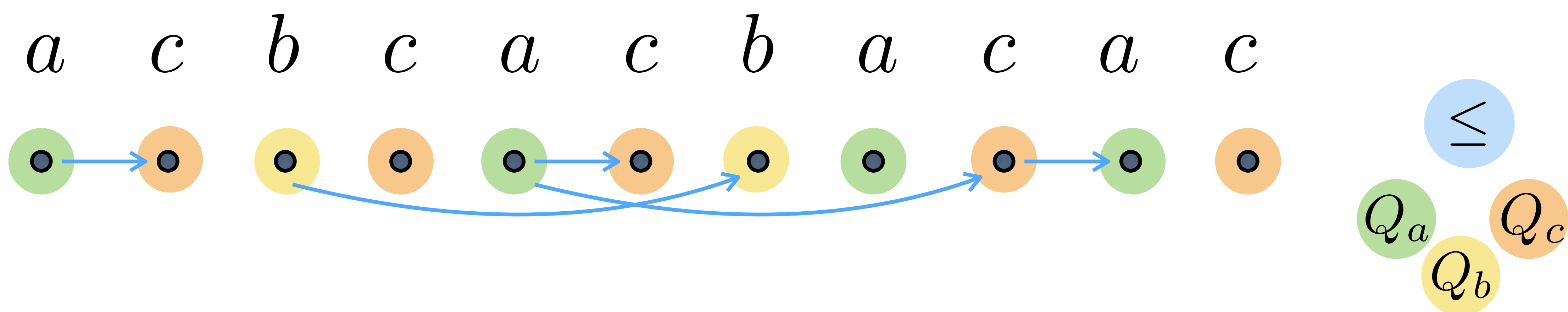

- examples:

$$\forall x. Q_a(x) \Rightarrow \exists y. x < y \wedge Q_c(y)$$

$$\begin{aligned}\exists X. &(\forall x\ \exists y\ y \leq x \wedge y \in X) \wedge \\ &(\forall x\ \exists y\ y \geq x \wedge y \in X) \wedge \\ &(\forall x\ \forall y\ (x < y \wedge \neg(\exists z\ x < z < y)) \Rightarrow (x \in X \Leftrightarrow y \notin X)).\end{aligned}$$

Our focus

MSO-definable

$x < y$ $\quad Q_a(x)$ $\quad x \in X$

$\phi \vee \psi$ $\quad \neg\phi$ $\quad \exists X.\phi$

$\Longleftrightarrow$

recognized by finite monoids

$$\begin{array}{cccc} \overleftarrow{h}(A) = & L & & A \\ & \cap\!| & & \cap\!| \\ & \Sigma^* & \xrightarrow[h]{} & M \end{array}$$

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⇐

- quite easy for finite words or trees
- difficult (or open) for other structures
- structure-specific arguments

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$\Leftarrow$

- quite easy for finite words or trees
- difficult (or open) for other structures
- structure-specific arguments

$\Rightarrow$

- relatively easy for all cases
- the arguments look generic

Our focus

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⇕

least class closed under:

- $0^*1^* \subseteq \{0,1\}^*$
- boolean combinations
- inv. images along $h : \Sigma \to \Gamma^*$
- dir. images along $h : \Sigma \twoheadrightarrow \Gamma$

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Definable implies recognizable, for finite words

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recognized by a finite monoid

$$\overleftarrow{h}(A) = L \subseteq \Sigma^* \xrightarrow{h} M \supseteq A$$

- $0^*1^* \subseteq \{0,1\}^*$ **recognized**
- L_i **rec. by** $h_i : \Sigma^* \to M_i$ **(for** $i = 1, 2$ **)**
  implies $L_1 \cap L_2$ **rec. by** $\langle h_1, h_2 \rangle : \Sigma^* \to M_1 \times M_2$

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  $\Sigma^* \setminus L_i$ rec. by h_i

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recognized by a finite monoid

$\overleftarrow{h}(A) = L$, $L \subseteq \Sigma^*$, $A \subseteq M$, $\Sigma^* \xrightarrow{h} M$

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- L_i rec. by $h_i : \Sigma^* \to M_i$ (for $i = 1, 2$)
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  $\Sigma^* \setminus L_i$ rec. by h_i
- L rec. by $h : \Gamma^* \to M$,
  implies $\overleftarrow{g}(L)$ rec. by $h \circ \hat{g}$

$g : \Sigma \to \Gamma^*$
$\hat{g} : \Sigma^* \to \Gamma^*$

The powerset construction

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- let $L \subseteq \Sigma^*$ be recognized by $h : \Sigma^* \to M$

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$$\overleftarrow{h}(A) = L \subseteq \Sigma^*, \quad A \subseteq M, \quad \Sigma^* \xrightarrow{h} M$$

- let $L \subseteq \Sigma^*$ be recognized by $h : \Sigma^* \to M$
- take $g : \Sigma \to \Gamma$
- define a monoid on $\mathcal{P}M$:

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  $B \subseteq \mathcal{P}M$ s.t. $B = \{S \mid S \cap A \neq \emptyset\}$
- then k and B recognize $g^*(L)$

We have just shown:

The class of languages
recognized by finite monoids
is closed under:

- boolean combinations
- inverse images along homomorphisms,
- direct images along (surjective) letter-to-letter homomorphisms.

Definable implies recognizable, for finite words

We have just shown:

The class of languages
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Does this work for other structures?

Monads

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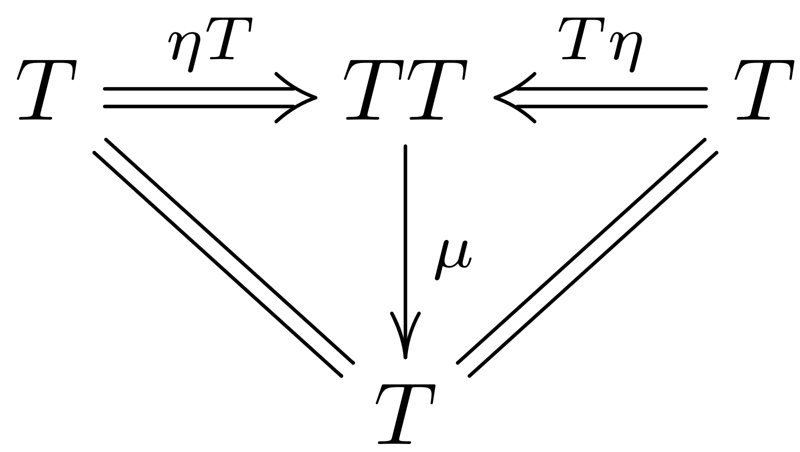

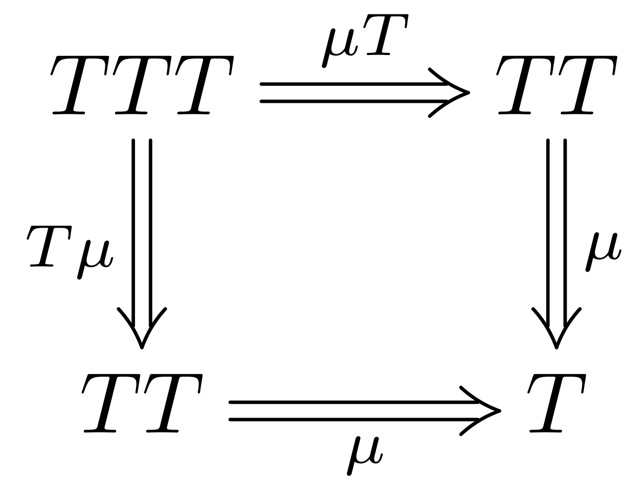

Monads

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$$\begin{array}{ccccc} T & \overset{\eta T}{\Longrightarrow} & TT & \overset{T\eta}{\Longleftarrow} & T \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} & \diagup\!\!\diagup & \\ & & T & & \end{array} \qquad\qquad \begin{array}{ccc} TTT & \overset{\mu T}{\Longrightarrow} & TT \\ {\scriptstyle T\mu}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ TT & \underset{\mu}{\Longrightarrow} & T \end{array}$$

Algebras: $a : TA \to A$

$$\begin{array}{ccc} A & \overset{\eta_A}{\longrightarrow} & TA \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad \begin{array}{ccc} TTA & \overset{\mu_A}{\longrightarrow} & TA \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \underset{a}{\longrightarrow} & A \end{array}$$

Monads

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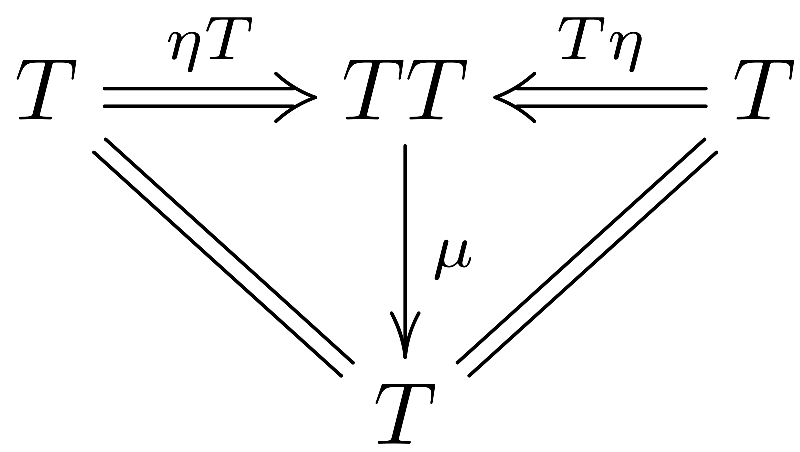

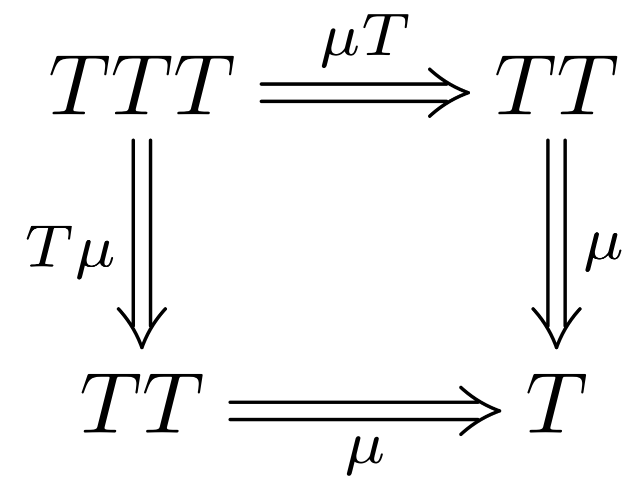

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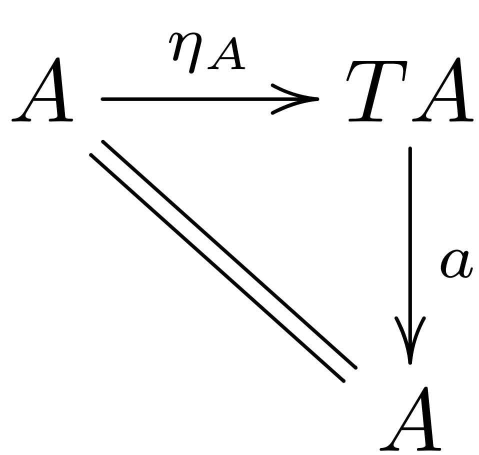
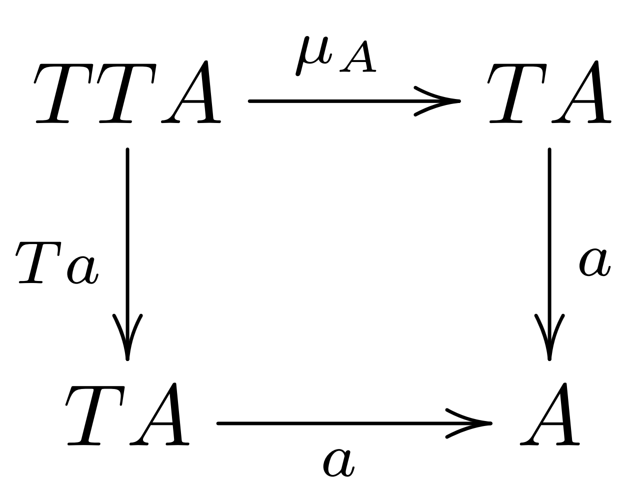
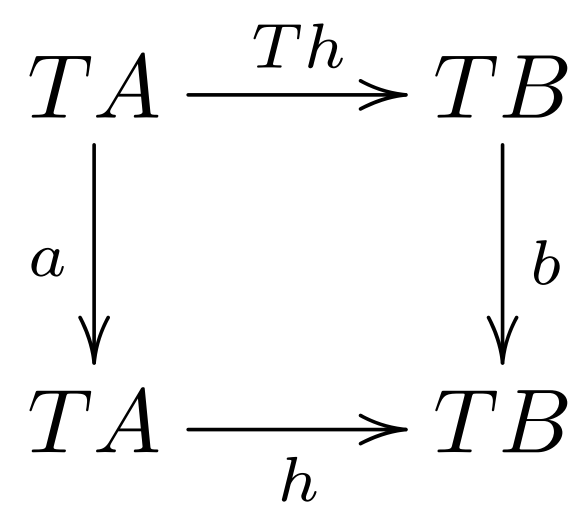




homomorphisms:



Examples

1. The list monad

$$TX = X^* \qquad Tf(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$$

$$\eta_X(x) = x \qquad \mu_X(w_1 w_2 \cdots w_n) = w_1 ^\frown w_2 ^\frown \cdots ^\frown w_n$$

algebras = monoids

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algebras = monoids

2. The powerset monad

$$TX = \mathcal{P}X \qquad Tf = \overrightarrow{f}$$

$$\eta_X(x) = \{x\} \qquad \mu_X(\Phi) = \bigcup \Phi$$

algebras = semilattices

3, 4, 5, ... : term monads

For an equational presentation (Σ, E) , put:

$TX = \Sigma$ -terms over X modulo the equations

Tf - variable substitution

η - variables as terms

μ - term flattening

algebras = (as expected)

Fact: $\mu_X : TTX \to TX$ is always a T-algebra.

Recognising languages with algebras

Fact: $\mu_X : TTX \to TX$ is always a T-algebra.

$$\begin{array}{ccc}
TT\Sigma & \xrightarrow{Th} & TM \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle m} \\
T\Sigma & \xrightarrow[h]{} & M \\
\cup\!| & & \cup\!| \\
L & & A
\end{array}$$

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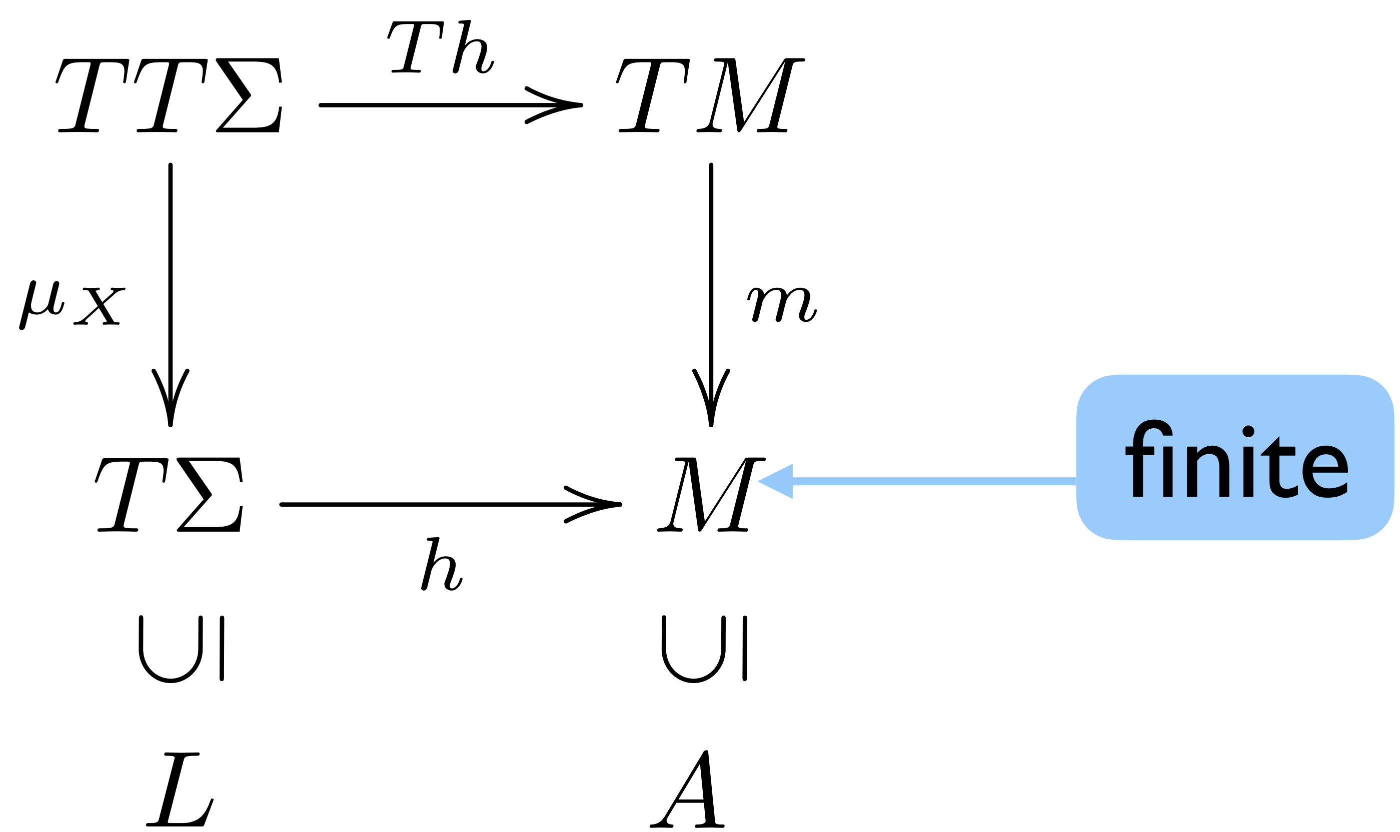

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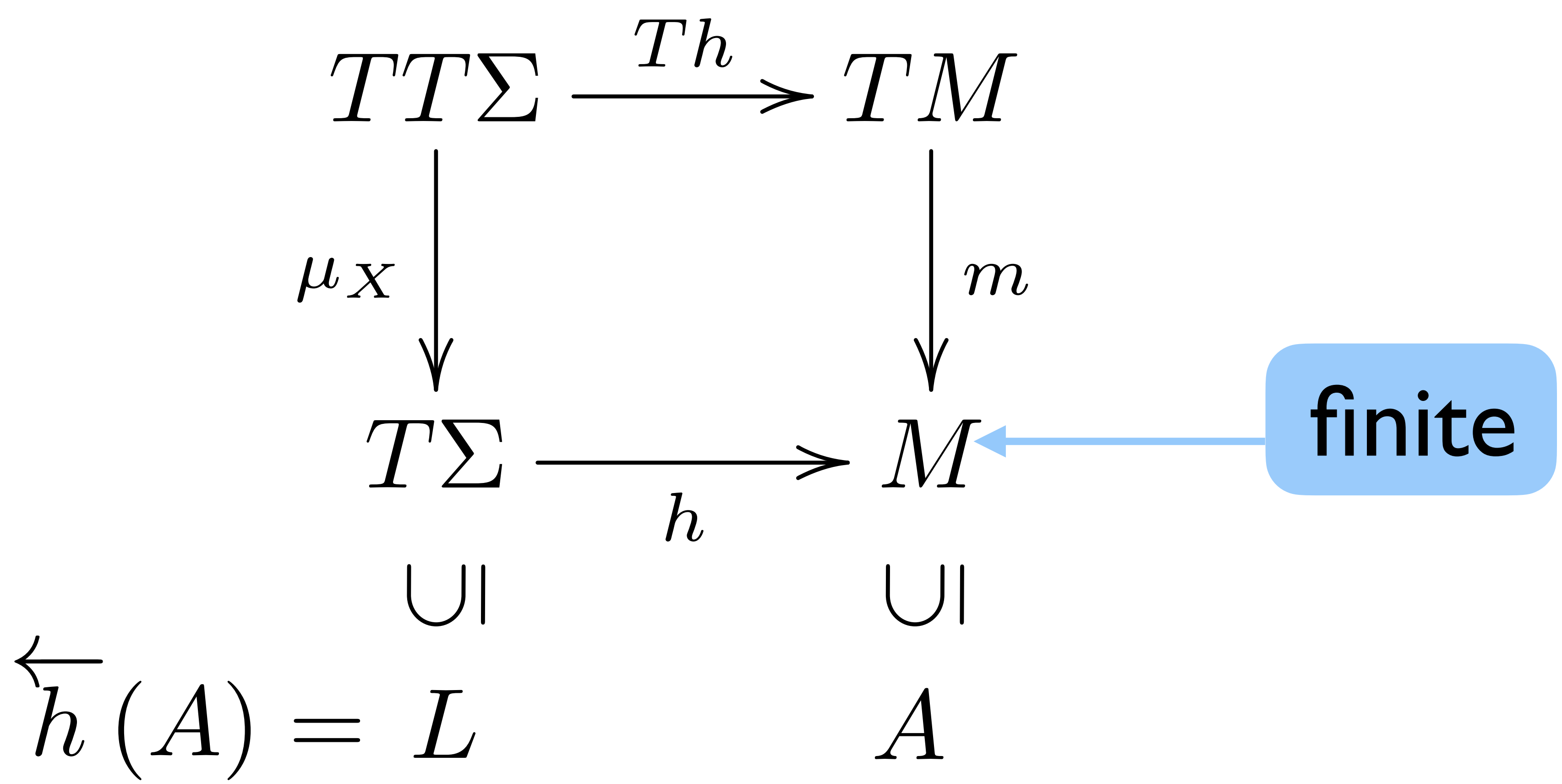
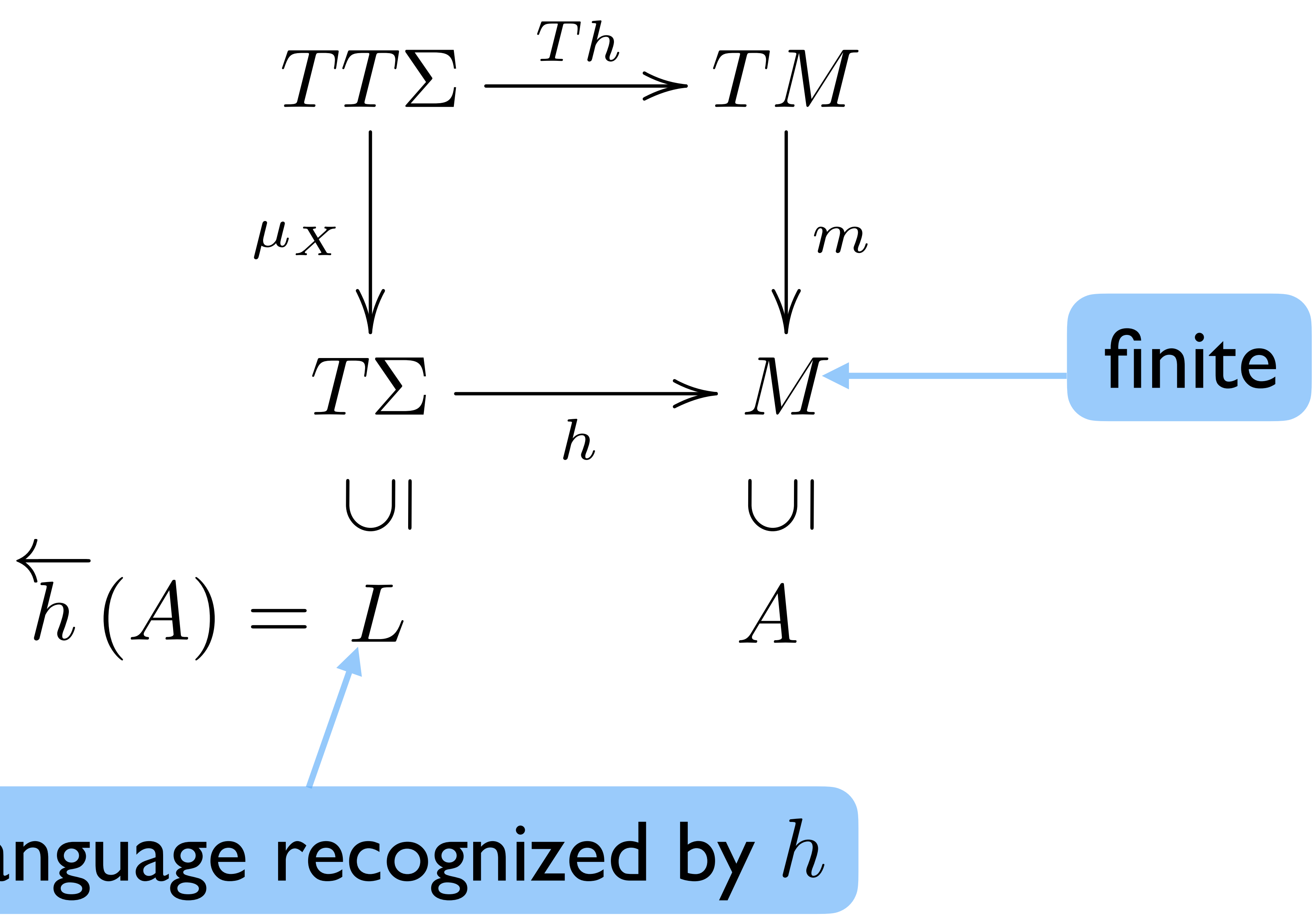


Recognising languages with algebras

Fact: $\mu_X : TTX \to TX$ is always a T-algebra.



The class of languages
recognized by finite algebras
is closed under:

- boolean combinations
- inverse images along homomorphisms,
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What we want to talk about

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Fact: $L \subseteq T\Sigma$ is recognizable iff
(the corresponding) $L \subseteq \Sigma^*$ is
regular and closed under W.

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Fact: $\overrightarrow{Th}(L)$ is not regular, so not T-recognizable.

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if: - T preserves weak pullbacks
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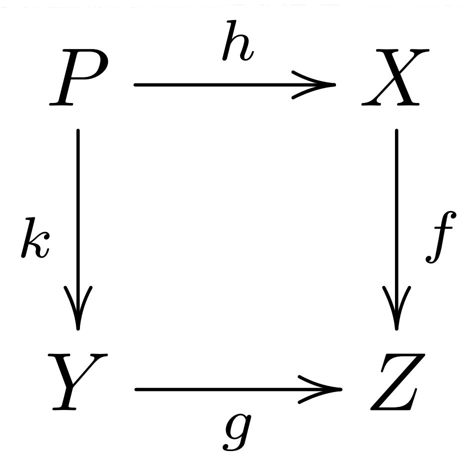
weak pullback:
for all $x \in X, y \in Y$ s.t. $f(x) = g(y)$
there is $p \in P$ s.t. $h(p) = x, k(p) = y$



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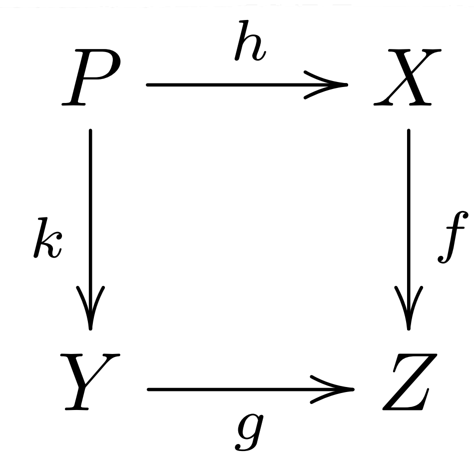
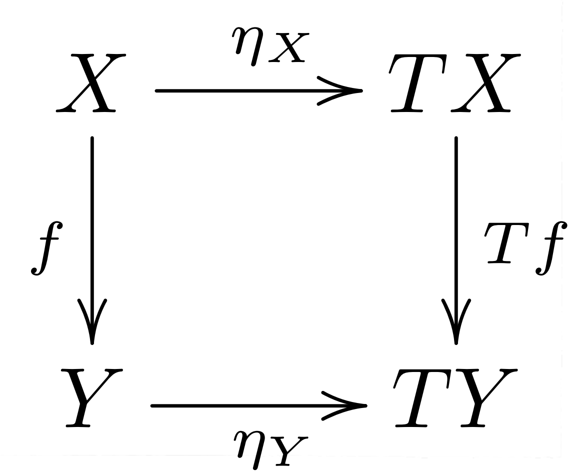


E.g. for η:

"a non-unit element never becomes a unit element after a substitution"



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Examples:

- any monad presented by *linear regular equations*:

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- T presented by a binary operation with:

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- define a T-algebra on $(\mathcal{P}A)^2$:

$$(\alpha_L, \alpha_R) \cdot (\beta_L, \beta_R) = \big(\{a \cdot b \mid a \in \alpha_L, b \in \beta_R\},$$
$$\{a_1 \cdot (a_2 \cdot (\cdots (a_n \cdot b) \cdot \cdot)) \mid n \geq 1, a_i \in \alpha_L, b \in \beta_R\}\big),$$

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A generalised powerset construction?

Sufficient condition II

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if it admits (an eq. presentation with)
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- Heyting algebras

$$t(x, y, z) = ((x \to y) \to z) \wedge ((z \to y) \to z) \wedge (x \vee z)$$

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4. The "almost Mal'cevian" monad: a ternary operation with

$$o(x, x, y) = o(y, x, x)$$

The landscape of monads