On Games over Algebras

Glynn Winskel. SmP, 27 June 2021

Proposes integration of descriptive compexity with a general theory of games which supports resource.

General reason: to take advantage of a *resourceful* model based on concurrent games and strategies, developed and well tested in semantics; it supports the computational, logical, quantitative aspects, so resource as number of pebbles, degree of parallelism, probabilistic and quantum resource, ...

Specific issues: Oddities, probable limitations, in presenting strategies as coKleisli maps, homomorphisms $T(A) \to B$: bias to one-way games ; composition of strategies = composition of coKleisli maps, is not obviously the usual composition of strategies!

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Games for interaction, via the composition of strategies

In 2-party games read Player vs. Opponent as Process vs. Environment.
Follow the paradigm of Conway, Joyal to achieve compositionality.

Assume operations on (2-party) games:

Dual game $G^{\perp}$ *- interchange the role of Player and Opponent;*
Counter-strategy = strategy for Opponent = strategy for Player in dual game.

Parallel composition of games $G \| H$.

A strategy (for Player) from a game G *to a game* H = *strategy in* $G^{\perp} \| H$.
A strategy (for Player) from a game H *to a game* K = *strategy in* $H^{\perp} \| K$.

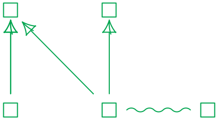
Compose by letting them play against each other in the common game H.

$\rightsquigarrow$ *a category with identity w.r.t. composition, the Copycat strategy in* $G^{\perp} \| G$, *so from* G *to* G *...*

Event structures - of the simplest kind

An event structure comprises $(E, \leqslant, \#)$, consisting of a set of events E
- partially ordered by $\leqslant$, the causal dependency relation, and
- a binary irreflexive symmetric relation, the conflict relation,
which satisfy $\{e' \mid e' \leqslant e\}$ is finite and $e \# e' \leqslant e'' \implies e \# e''$.

Two events are concurrent when neither in conflict nor causally related.



(drawn immediate conflict, and causal dependency)

The configurations of an event structure E consist of those subsets $x \subseteq E$ which are
Consistent: $\forall e, e' \in x.\ \neg(e \# e')$ *and*
Down-closed: $\forall e, e'.\ e' \leqslant e \in x \implies e' \in x$.

Event-structure game w.r.t. a signature

A signature (Σ, C, V) comprises Σ a many-sorted relational signature including equality; a set C event-name constants; a set $V = \{\alpha, \beta, \gamma, \cdots\}$ of variables.

A (Σ, C, V)-game comprises an event structure $(E, \leqslant, \#)$
– its moves are the events E, with

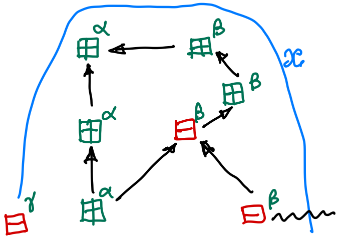
a polarity function $\mathrm{pol} : E \to \{+, -\}$ s.t. no immediate conflict $\boxplus \rightsquigarrow \boxminus$

a variable/constant assignment $\mathrm{var} : E \to C \cup V$ s.t.

$$e \; co \; e' \Rightarrow \mathrm{var}(e) \neq \mathrm{var}(e')$$

a winning condition WC, an assertion in the free logic over (Σ, C, V).

WC:
$\mathbb{E}(\gamma) \to \exists \beta.\ P(\alpha, \beta) \wedge Q(\beta)$
Existence predicate involves latest occurrence of variable in a configuration



A good reference for free logic: Dana Scott, Identity and Existence. LNM 753, 1979

Games over an algebra

A game over an algebra $(G, \mathcal{A})$ *comprises a* (Σ, C, V)*-game* G*, a* Σ*-algebra* $\mathcal{A}$*. It determines a (traditional) concurrent game* $expn(G, \mathcal{A})$ *in which each move with a variable* $\Box^{\alpha}$ *is expanded to its instances* $\Box^{a_1} \sim\!\sim\!\sim \Box^{a_2} \sim\!\sim\!\sim \cdots$

A strategy σ *in* $(G, \mathcal{A})$ *assigns values in* $\mathcal{A}$ *to Player moves of the game* G *in answer to assignments of Opponent. Described as a map of event structures, it corresponds to a (traditional) concurrent strategy* σ' *in* $expn(G, \mathcal{A})$*:*

$$\begin{tikzcd} S \arrow[r, dotted, "\sigma'"] \arrow[dr, "\sigma"'] & expn(G, \mathcal{A}) \arrow[d] \\ & G \end{tikzcd}$$

For a configuration x *of* S *and a* Σ*-assertion* φ*,*
$x \models \varphi$ *will mean latest assignments to variables in* x *make* φ *true.*
The strategy is winning means $x \models WC$ *for all* $+$*-maximal configs* x *of* S*.*

Proposition. The events S of a strategy form a Σ-algebra:

$$R_S(s_1, \cdots, s_n) \text{ iff } x \models R(\mathrm{var}(\sigma(s_1)), \cdots, \mathrm{var}(\sigma(s_n)))\,,$$

for some configuration x of S with $s_1, \cdots, s_n \in x$.

Corollary. $(G, \mathcal{A})$ determines a canonical Σ-algebra, on events of $expn(G, \mathcal{A})$.

Constructions on games

Let G be a (Σ, C, V)-game. Its dual $G^{\perp}$ is the (Σ, C, V)-game obtained by reversing polarities, i.e. the roles of Player and Opponent, with winning condition $\neg WC_G$.

Let G be a (Σ_G, C_G, V_G)-game. Let H be a (Σ_H, C_H, V_H)-game. Their parallel composition $G \| H$ is the $(\Sigma_G + \Sigma_H, C_G + C_H, V_G + V_H)$-game comprising the parallel juxtaposition of event structures with winning condition $WC_G \vee WC_H$.

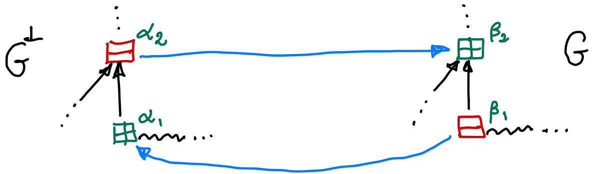
Let $(G, \mathcal{A})$ to $(H, \mathcal{B})$ be games over algebras. A winning strategy from $(G, \mathcal{A})$ to $(H, \mathcal{B})$, $\sigma : (G, \mathcal{A}) \mathrel{+\!\!\!\!\rightarrow} (H, \mathcal{B})$, comprises a winning strategy in the game $(G^{\perp} \| H, \mathcal{A} + \mathcal{B})$ - its winning condition is $WC_G \rightarrow WC_H$.

Theorem. *Obtain a category $\mathcal{G}ames$ of winning strategies between games over algebras: winning strategies compose with the copycat strategy as identity.*

$\rightsquigarrow$ Reductions: a winning strategy $\sigma : (G, \mathcal{A}) \mathrel{+\!\!\!\!\rightarrow} (\mathcal{H}, \mathcal{B})$ reduces the problem of finding a winning strategy in $(H, \mathcal{B})$ to finding a winning strategy in $(G, \mathcal{A})$. A winning strategy in $(G, \mathcal{A})$ is a winning strategy $(\emptyset, \emptyset) \mathrel{+\!\!\!\!\rightarrow} (G, \mathcal{A})$; its composition with σ is a winning strategy in $(H, \mathcal{B})$.

Duplicator-Spoiler games deconstructed

Let G be a (Σ, C, V)-game with winning condition WC.
Its Duplicator-Spoiler game *is a $(\Sigma + \Sigma, C + C, V + V)$-game with event structure:*



Its winning condition is $WC_1 \to WC_2$
α β
(renaming variables WC to the appropriate component of $V + V$).

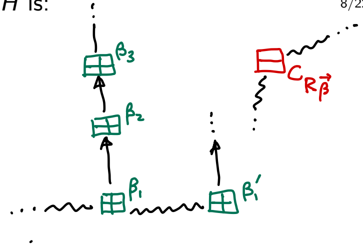
A Duplicator-Spoiler strategy*, for G, is a winning deterministic strategy*

$$\sigma : (G, \mathcal{A}) \twoheadrightarrow\!\!\!\!\!\!\!\!\;\!\!\!\!+\;\;\, (G, \mathcal{B})$$

with causal dependencies those of the Duplicator-Spoiler game.
Duplicator-Spoiler strategies compose, with identity the copycat strategy.

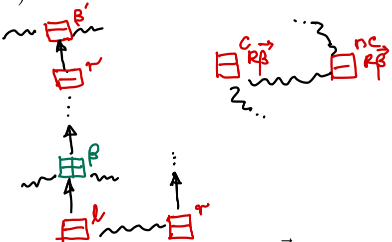
$\rightsquigarrow$ Duplicator-Spoiler category *$\mathcal{DS}(G)$, a subcategory of $\mathcal{G}ames$.*

Example, **Homomorphism games** $\mathcal{DS}(H)$ where H is:



with winning condition $\bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \to R(\vec{\beta})$ where $\vec{\beta}$ is a tuple of variables.

Example, **Ehrenfeucht-Fraïssé games** $\mathcal{DS}(EF)$ where EF is:



with winning condition $(\bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \to R(\vec{\beta})) \wedge (\bigwedge_{R\vec{\beta}} \mathbb{E}(nc_{R\vec{\beta}}) \to \neg R(\vec{\beta}))$.

Obtain pebbled versions by restricting variable sets.

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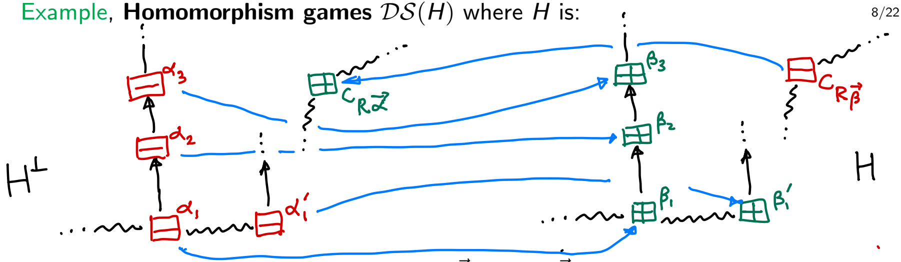

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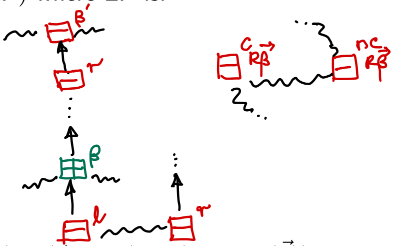

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Obtain pebbled versions by restricting variable sets.

An adjunction

Let G be a (Σ, C, V)-game. G is one-way if

$$\mathrm{var}(e) \in V \implies \mathrm{pol}(e) = + \,,$$

for all moves e. H is a one-way game, EF is not.

Proposition. *Let G be a one-way (Σ, C, V)-game with existential positive winning condition (which can include $(\mathbb{E}(\boxminus^c) \to \phi)$, where ϕ is existl. positive). There is a functor $R(G) : \Sigma\text{-}Alg \to \mathcal{DS}(G)$.*

Proposition. *The functor $R(H)$ has left adjoint expn :*

$$\frac{expn(H, \mathcal{A}) \;\to\; \mathcal{B}}{(H, \mathcal{A}) \twoheadrightarrow (H, \mathcal{B})}$$

The pebbling comonad arises from the adjunction: $T_k(\mathcal{A}) \cong expn(H, \mathcal{A})$, where k is the size of the set of variables V_H ; coKleisli$(T_k) \cong \mathcal{DS}(H)$.

There should be analogous adjunctions for "two-way" games involving algebras and pairs of homomorphisms $\mathcal{A} \rightleftarrows \mathcal{B}$