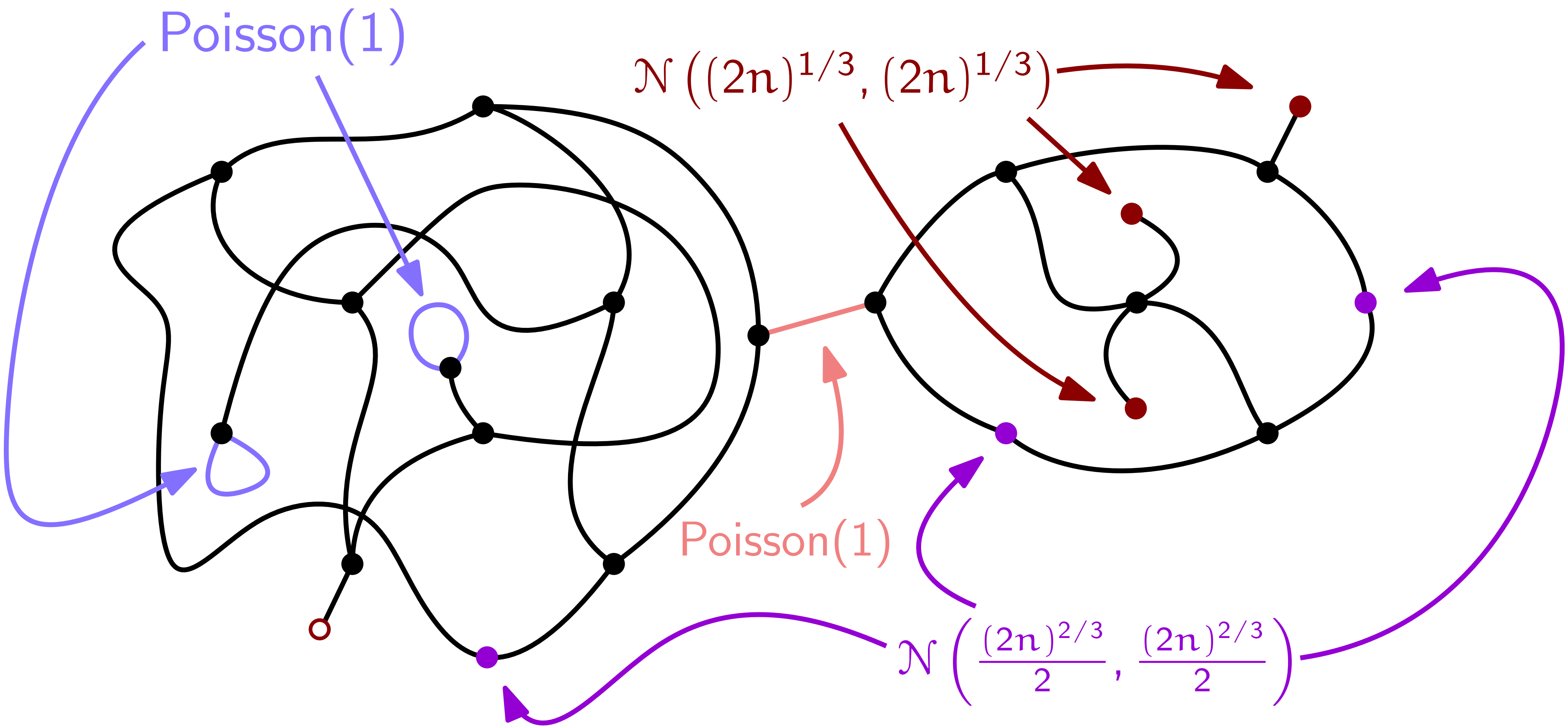
Distributions of parameters in restricted classes of maps and λ-terms



Structure Meets Power Workshop, LICS, 28 June 2021

Olivier Bodini (LIPN, Paris 13)
Alexandros Singh (LIPN, Paris 13)
Noam Zeilberger (LIX, Polytechnique)

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not in this talk!

Techniques drawn from combinatorics, logic, and physics may be used in tandem to study them!

Subfamilies of λ-terms

General terms: no restrictions on variable use

$\lambda x.\lambda y.x$

$\lambda x.\lambda y.x\ (y\ a)$

$(\lambda x.xx)(\lambda y.yy)$

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Affine Terms: bound variables occur **at most** once

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Linear Terms: bound variables occur **exactly** once

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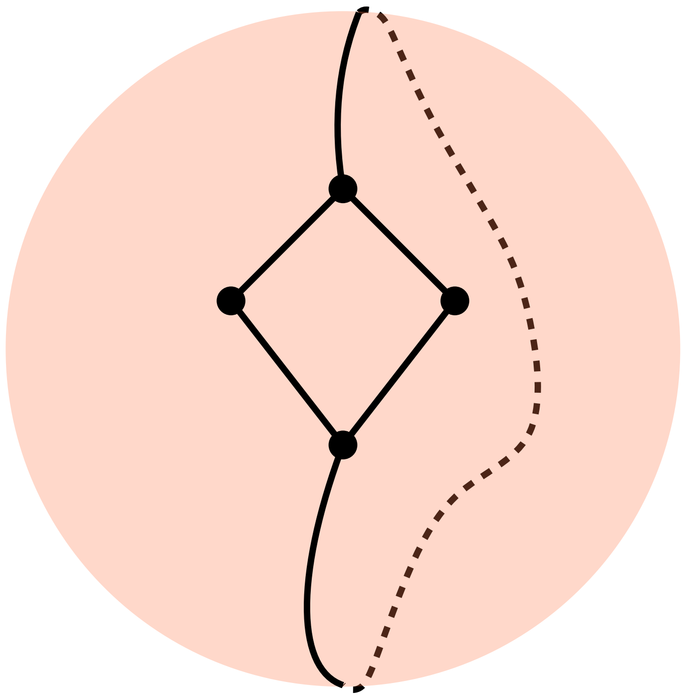
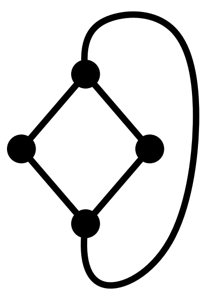
$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

What are maps?





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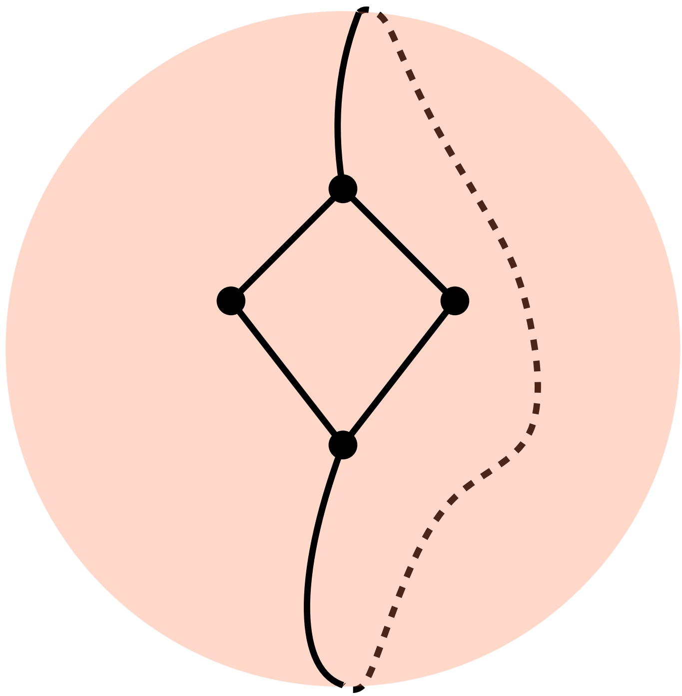

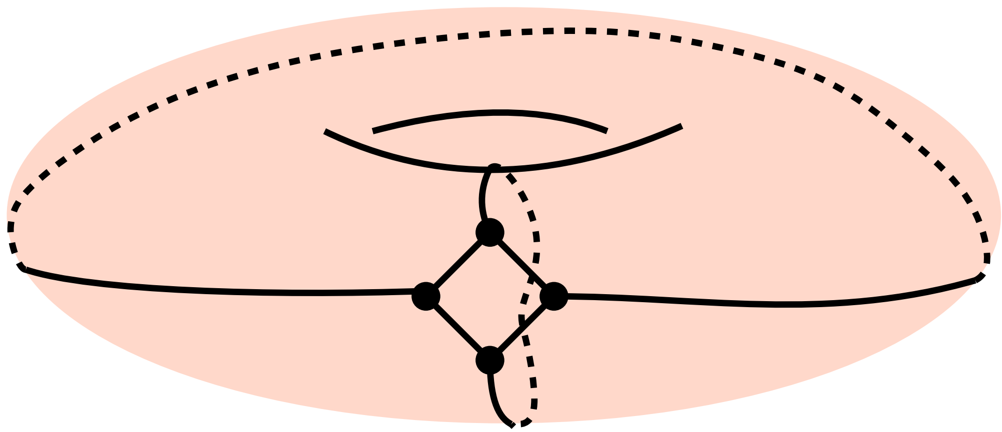

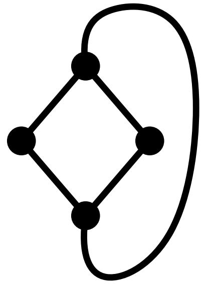

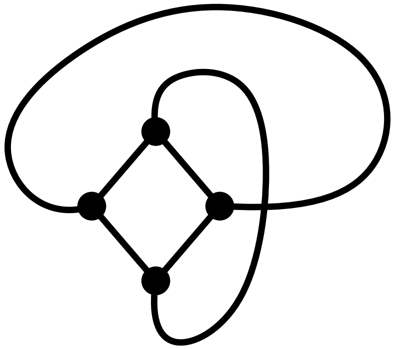

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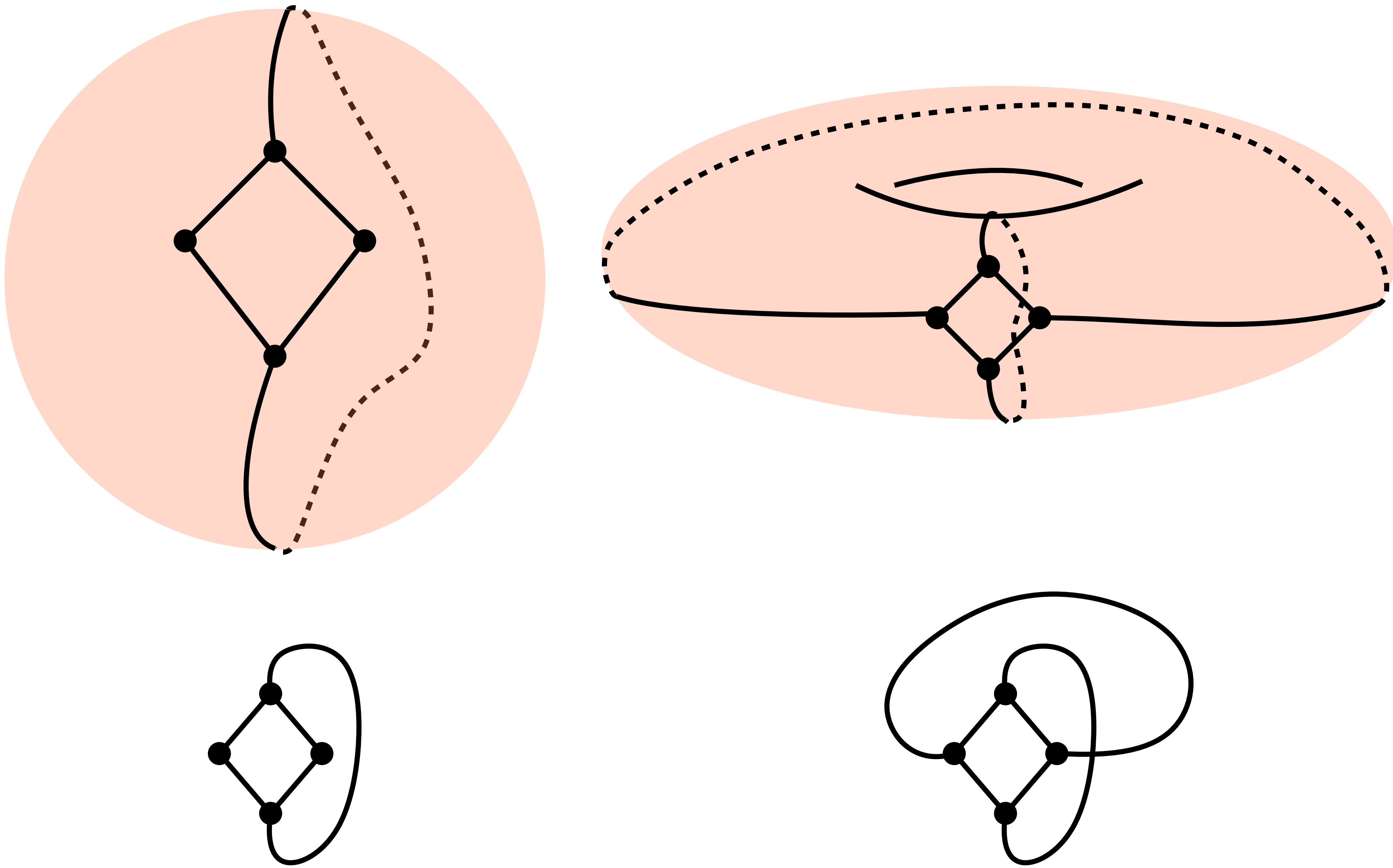
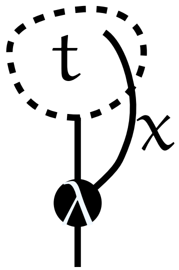


We're interested in unrestricted genus, restricted vertex degrees

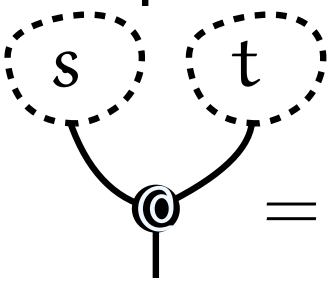
Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

$\bullet = x$



$= \lambda x.t$



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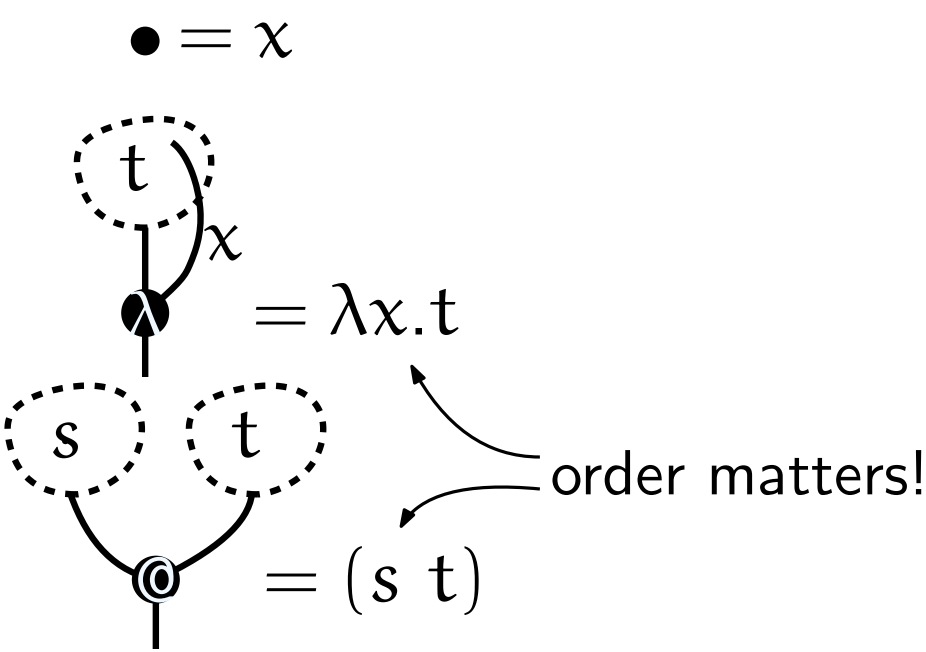

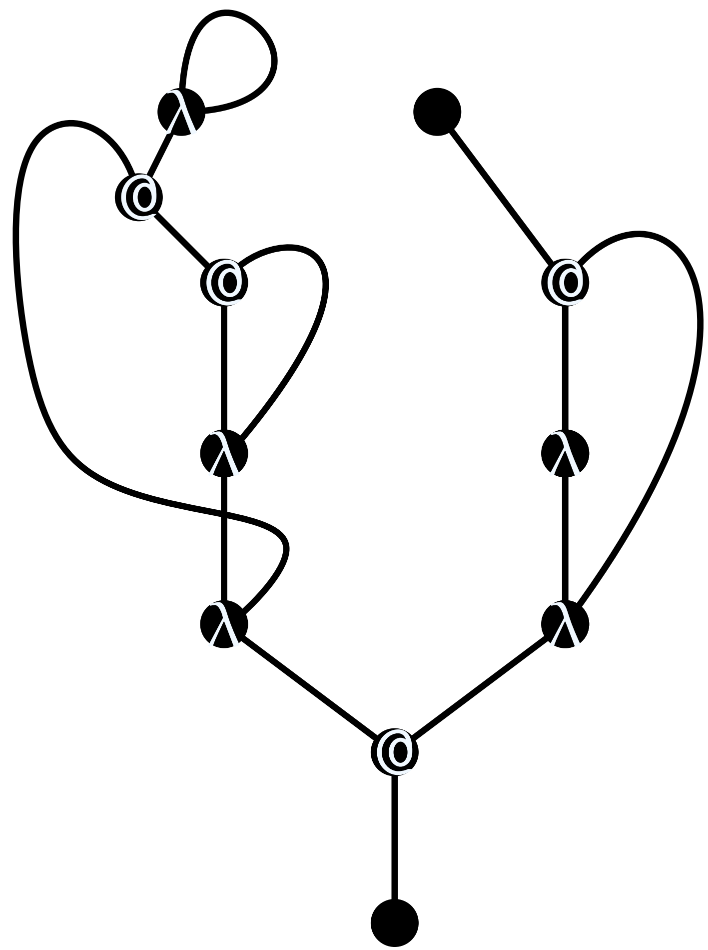

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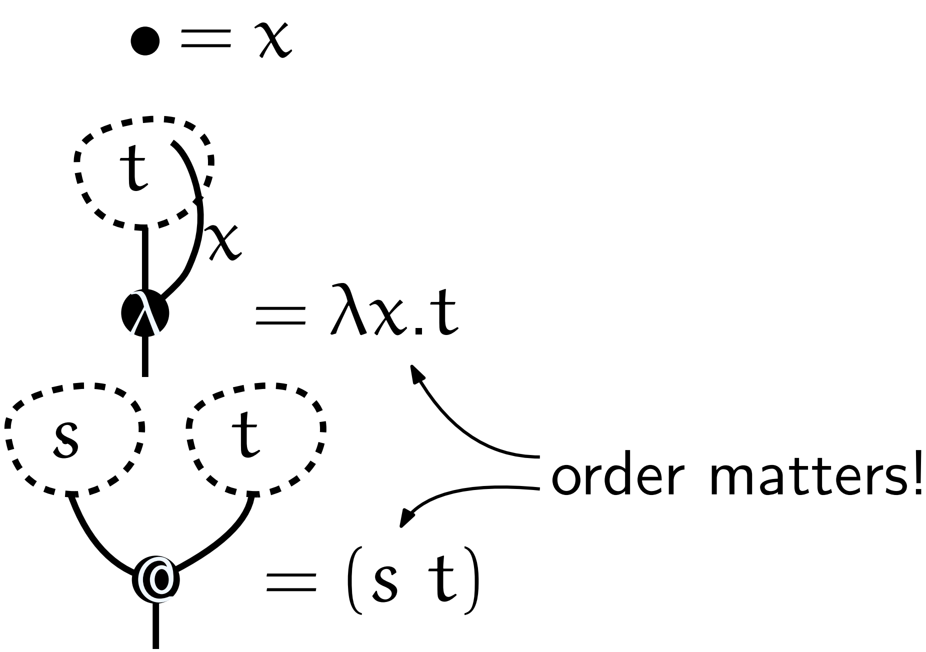

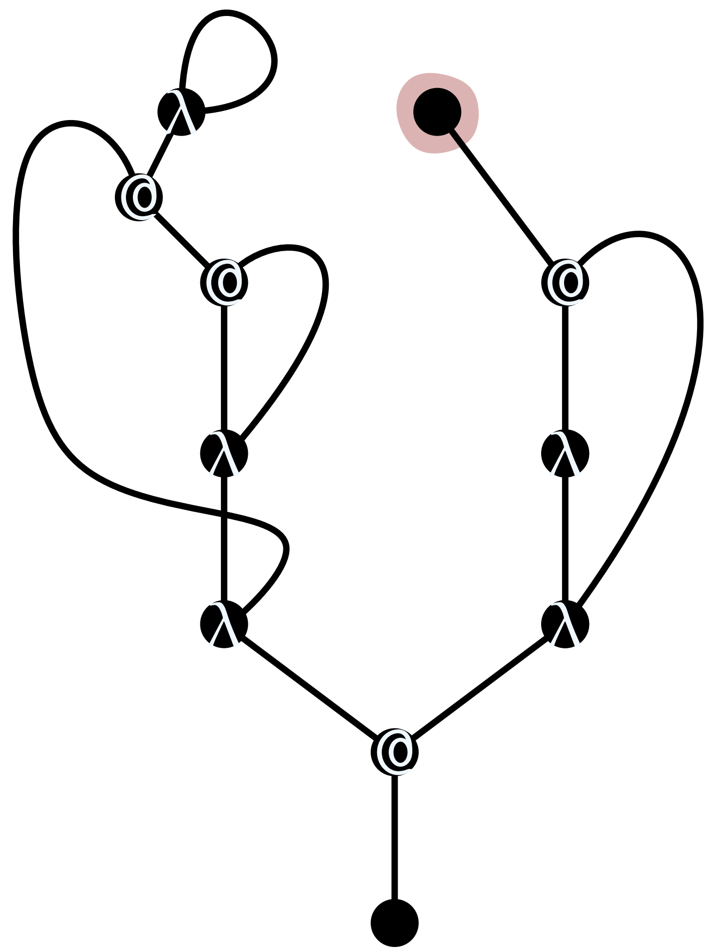

Dictionary

- Free var ↔ unary vertex

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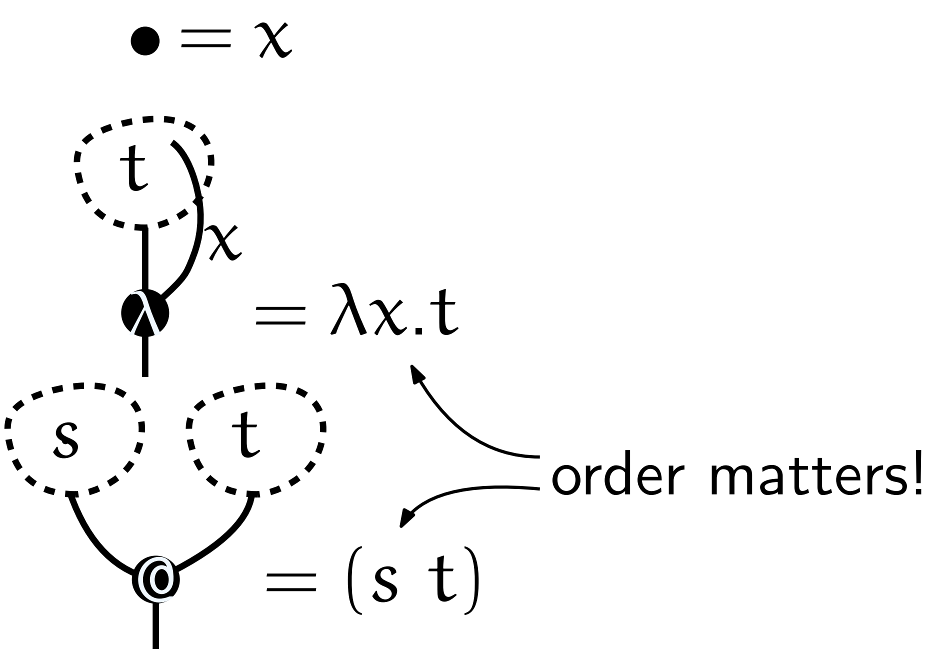

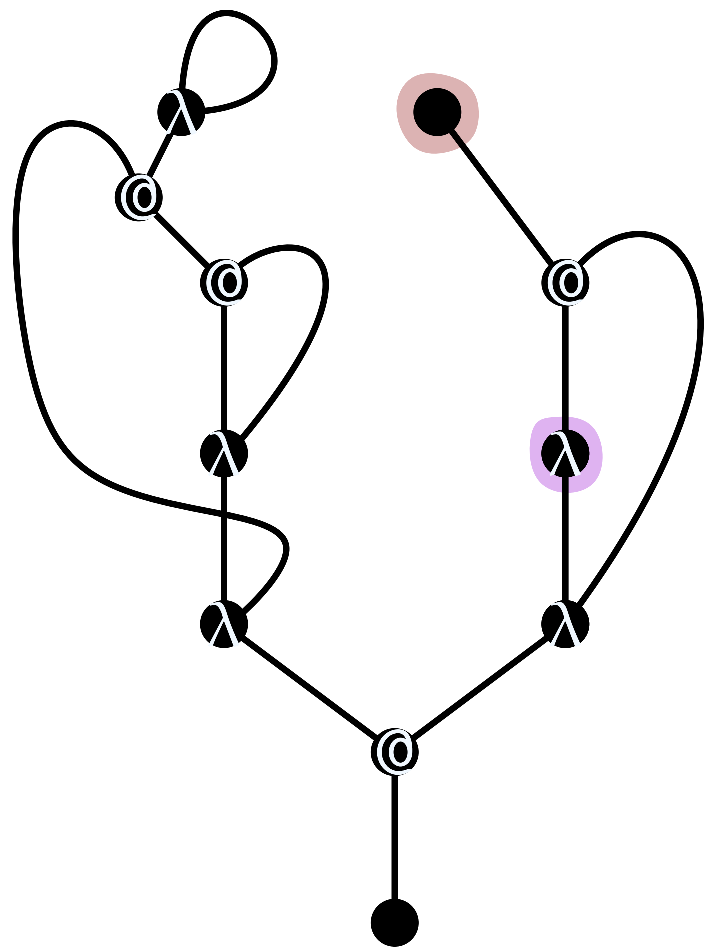

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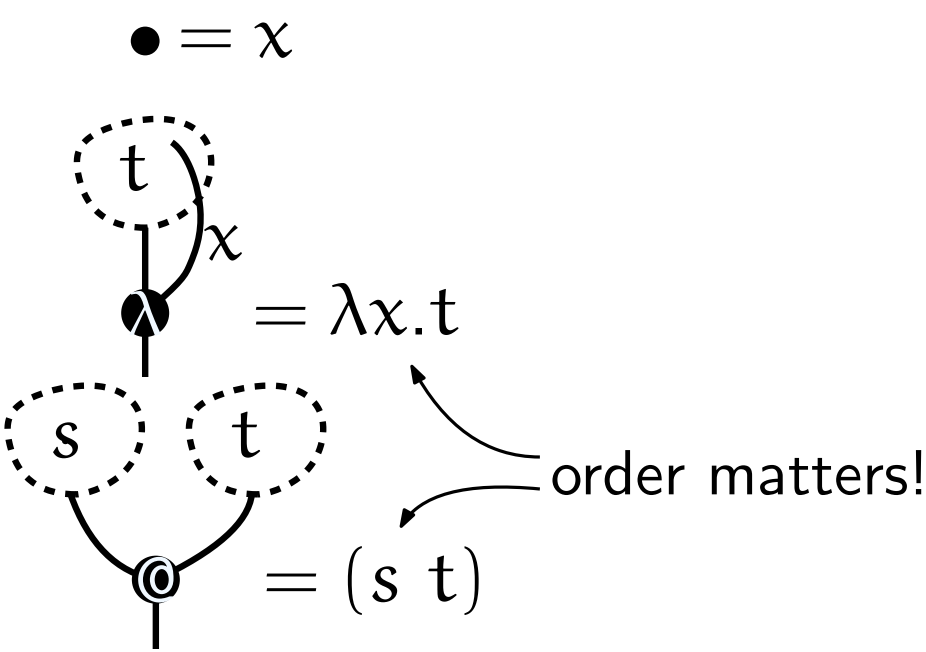

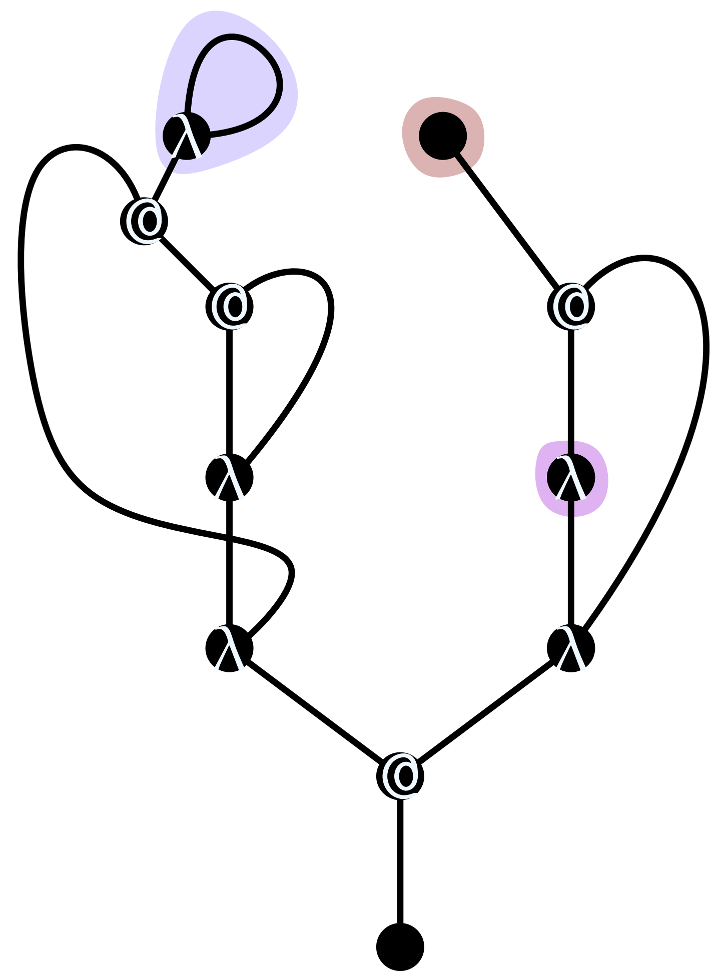

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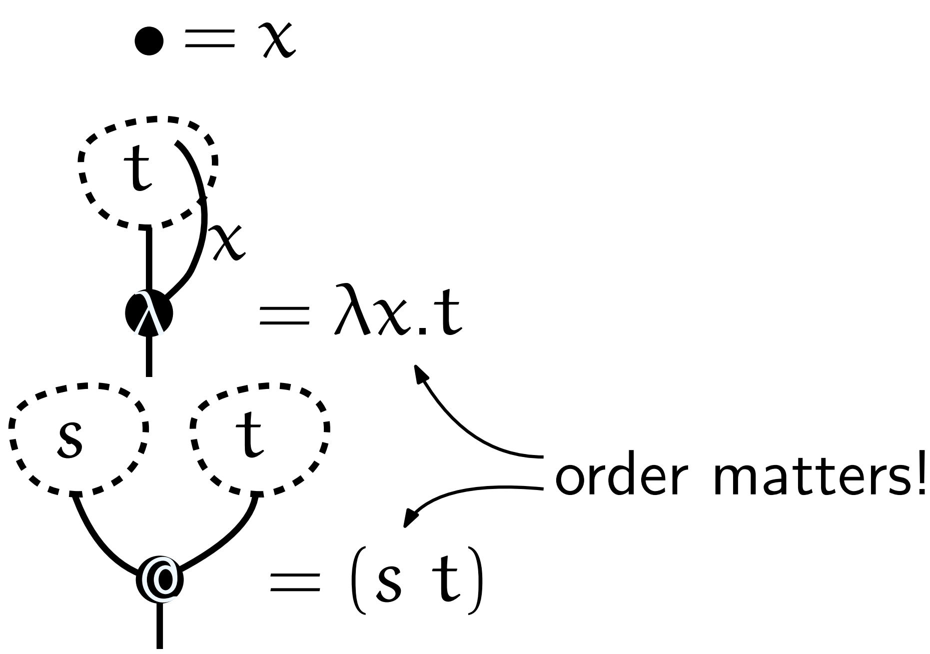

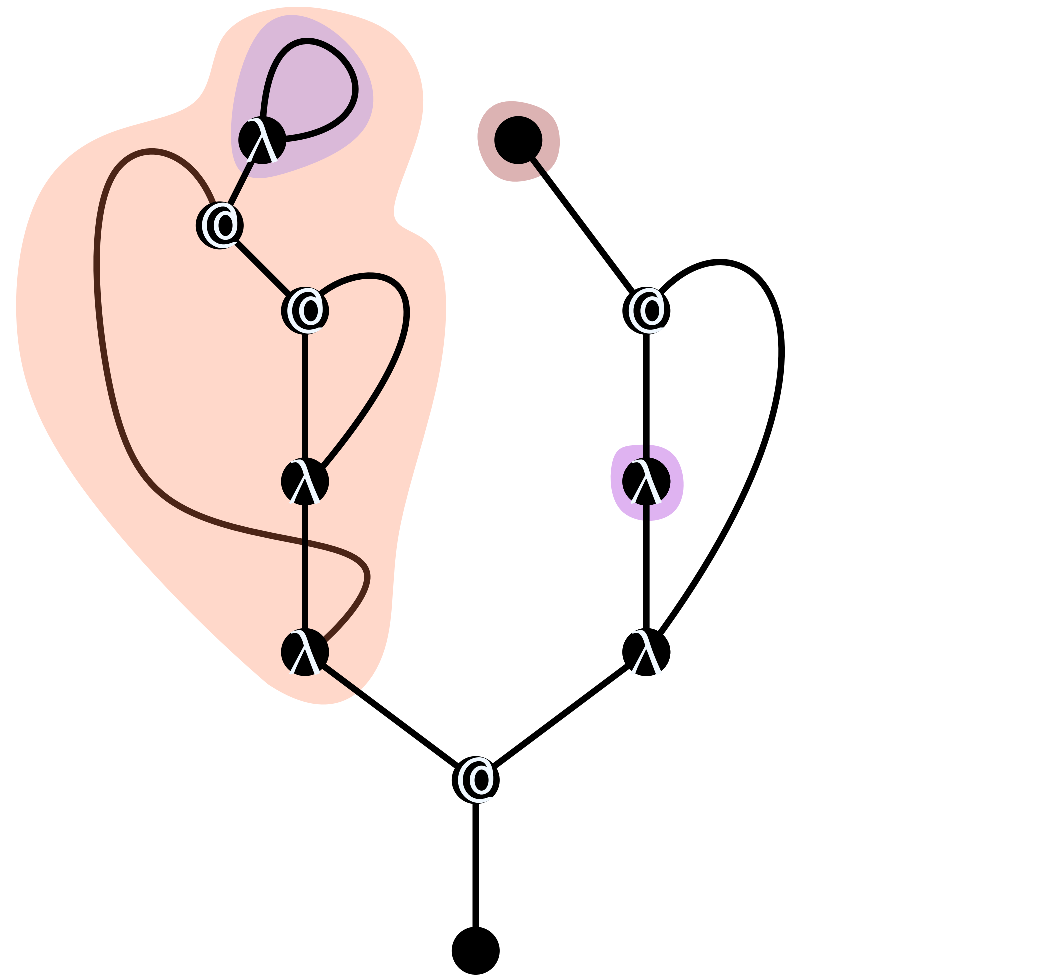

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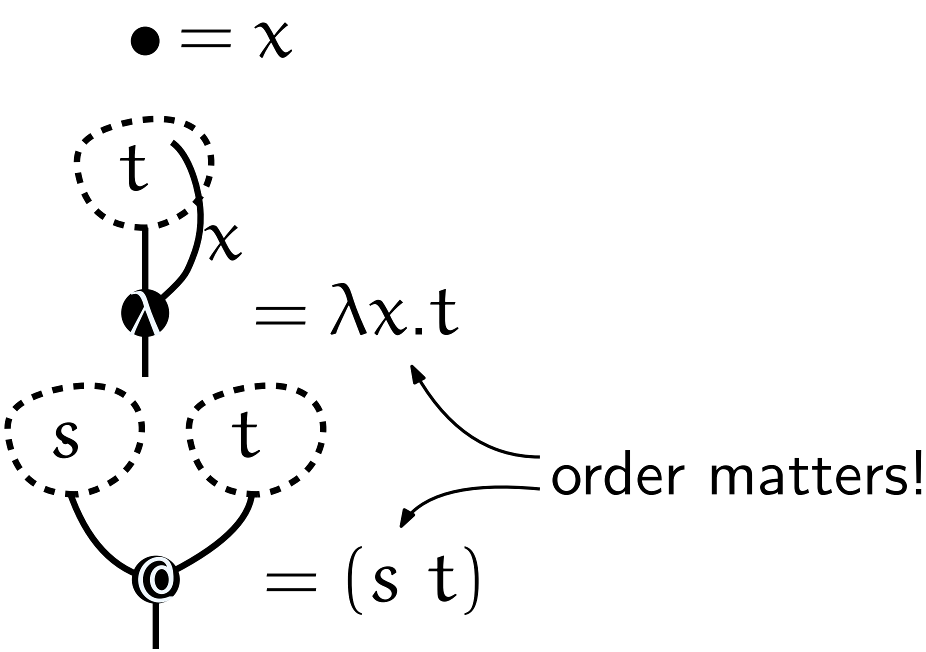

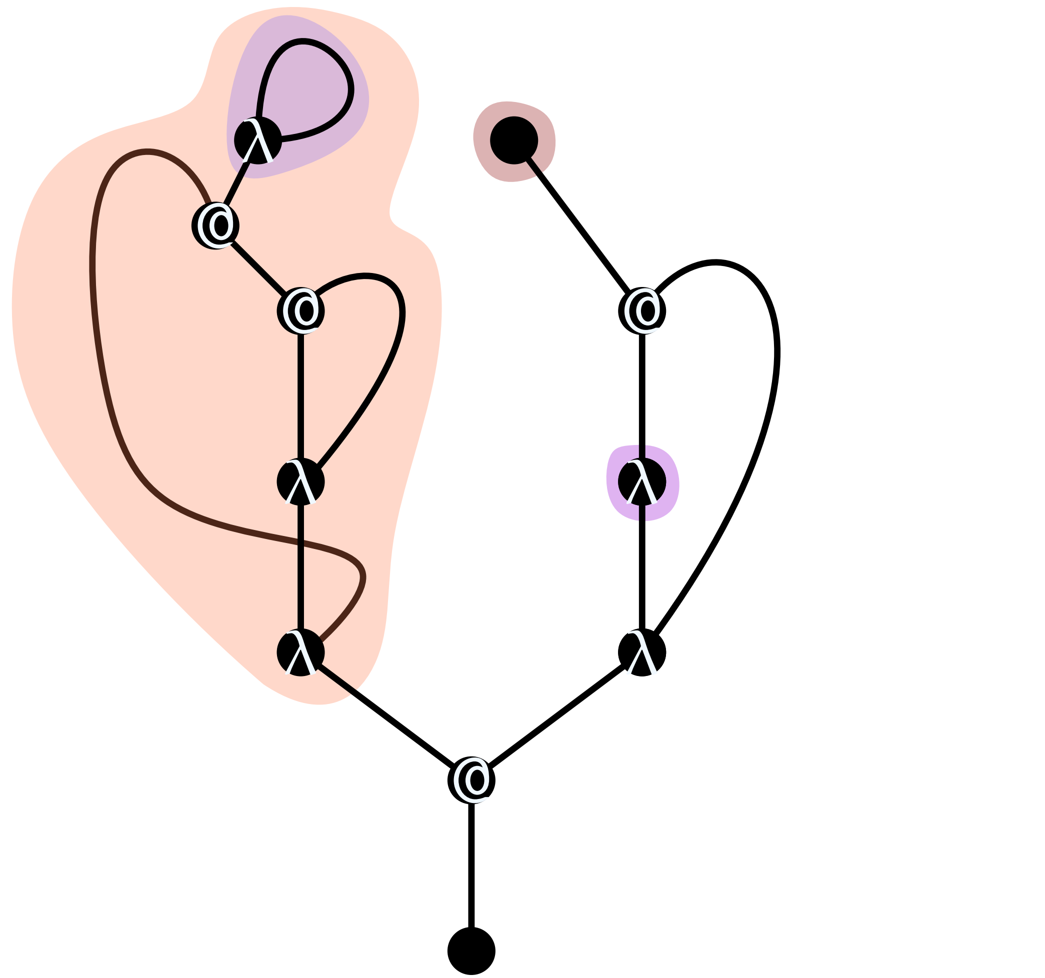

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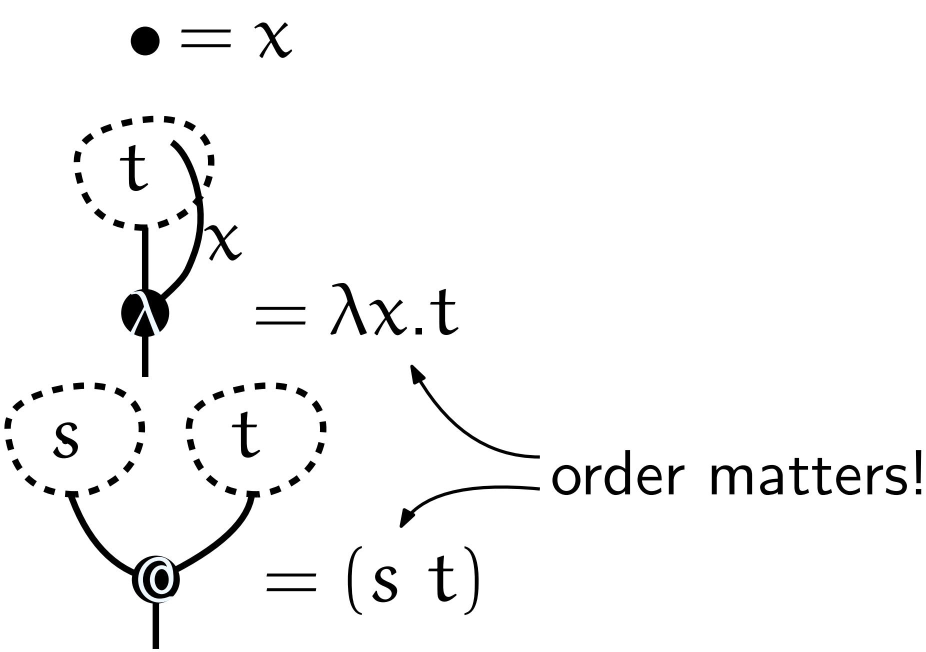

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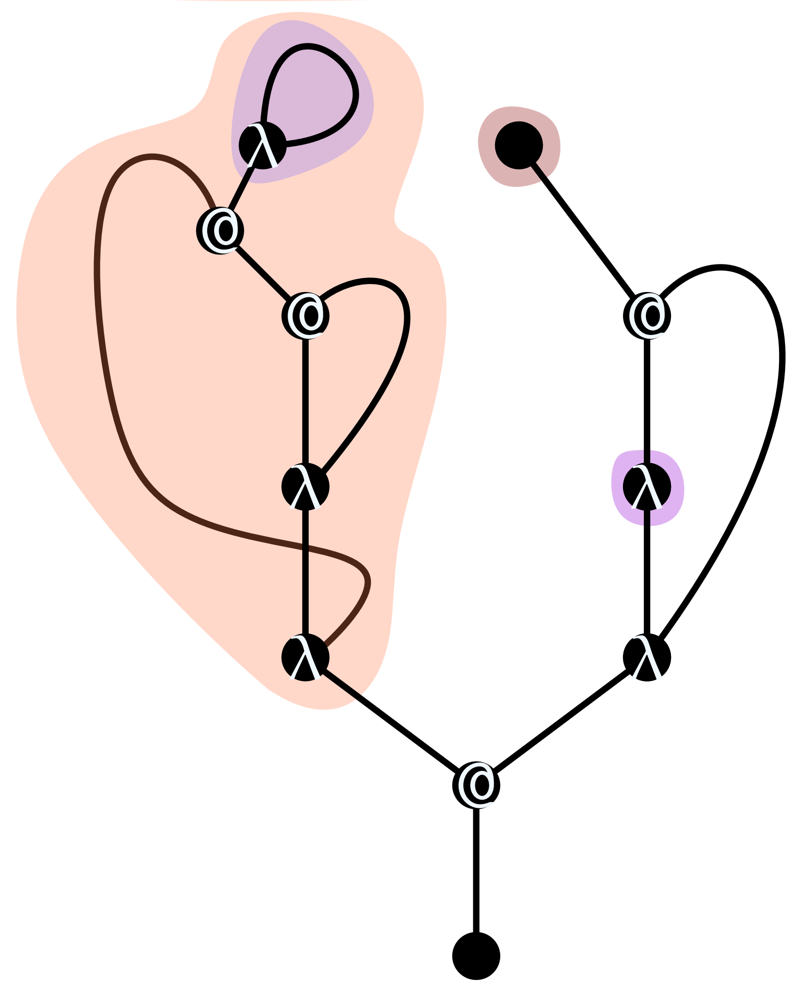

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Closed linear terms ↔ trivalent maps
Closed affine terms ↔ (2,3)-valent maps
Established in [BGJ13, BGGJ13]

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Decomposing open rooted trivalent maps à la Tutte [AB00]

edges → z; unary vertices → u

$$T(z, u) =$$

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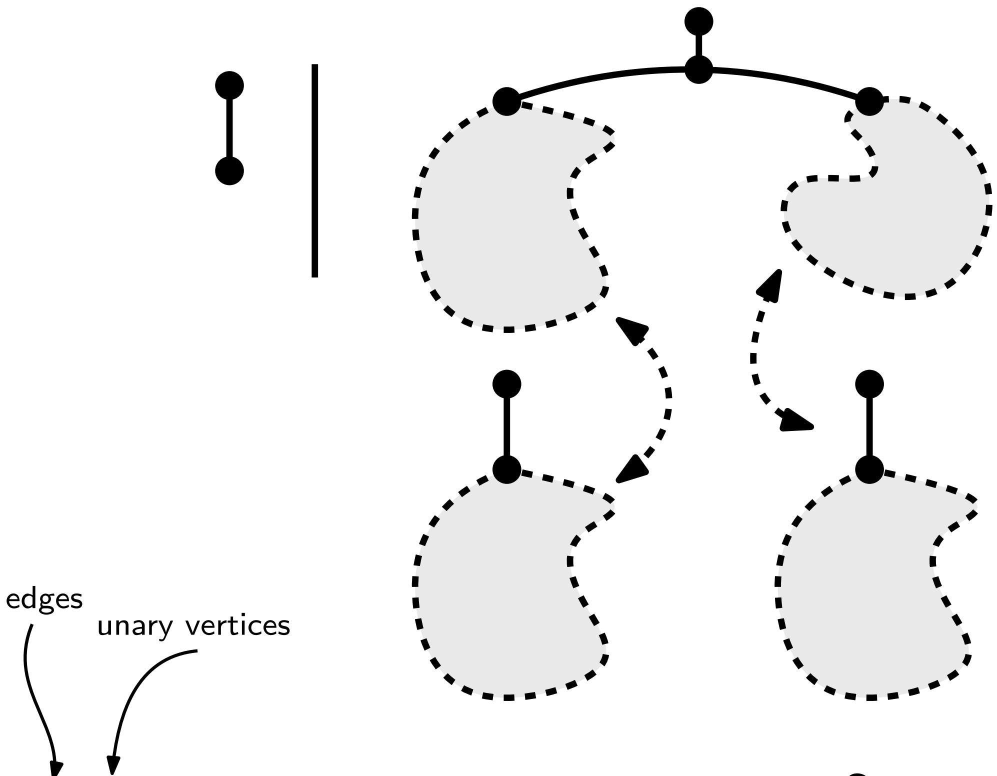

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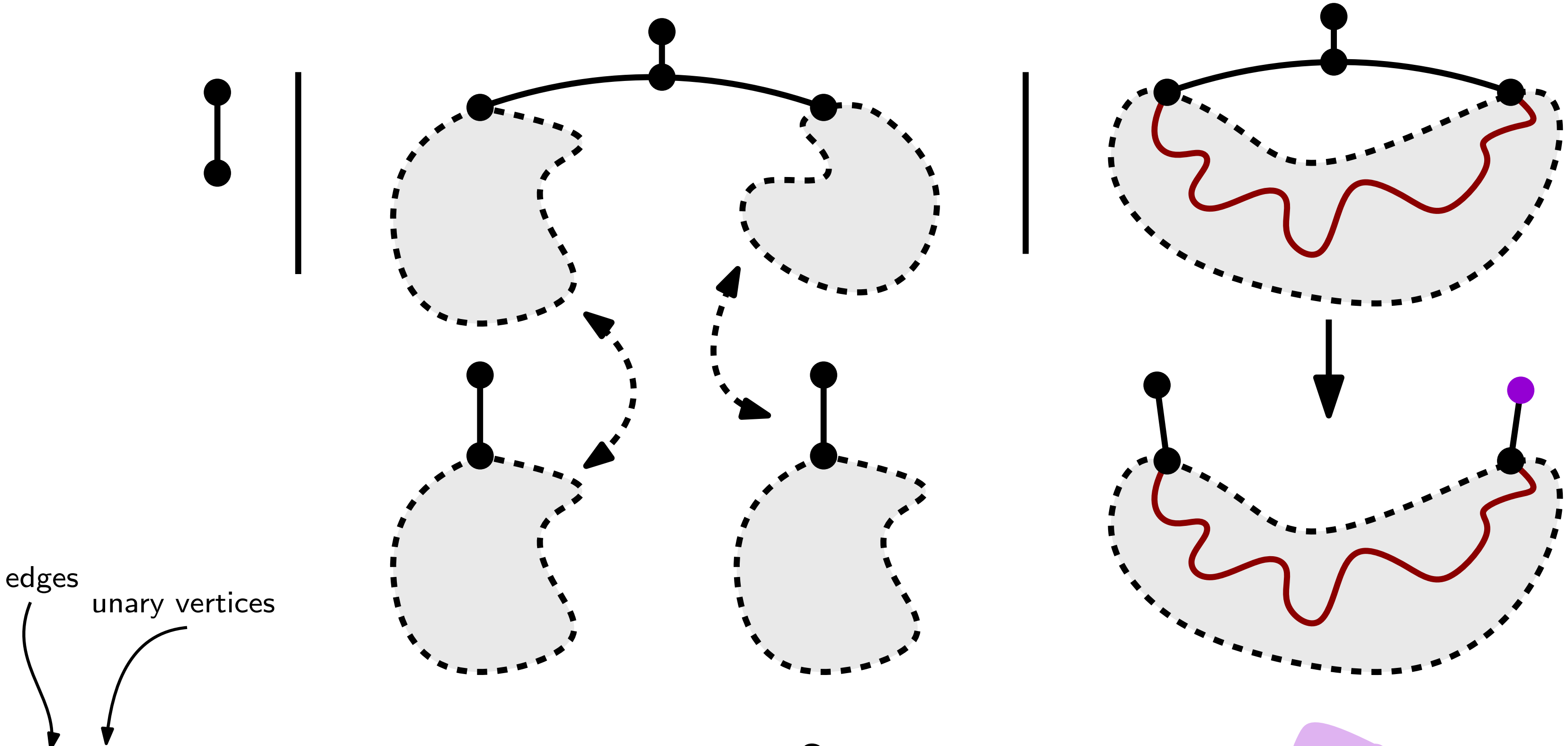

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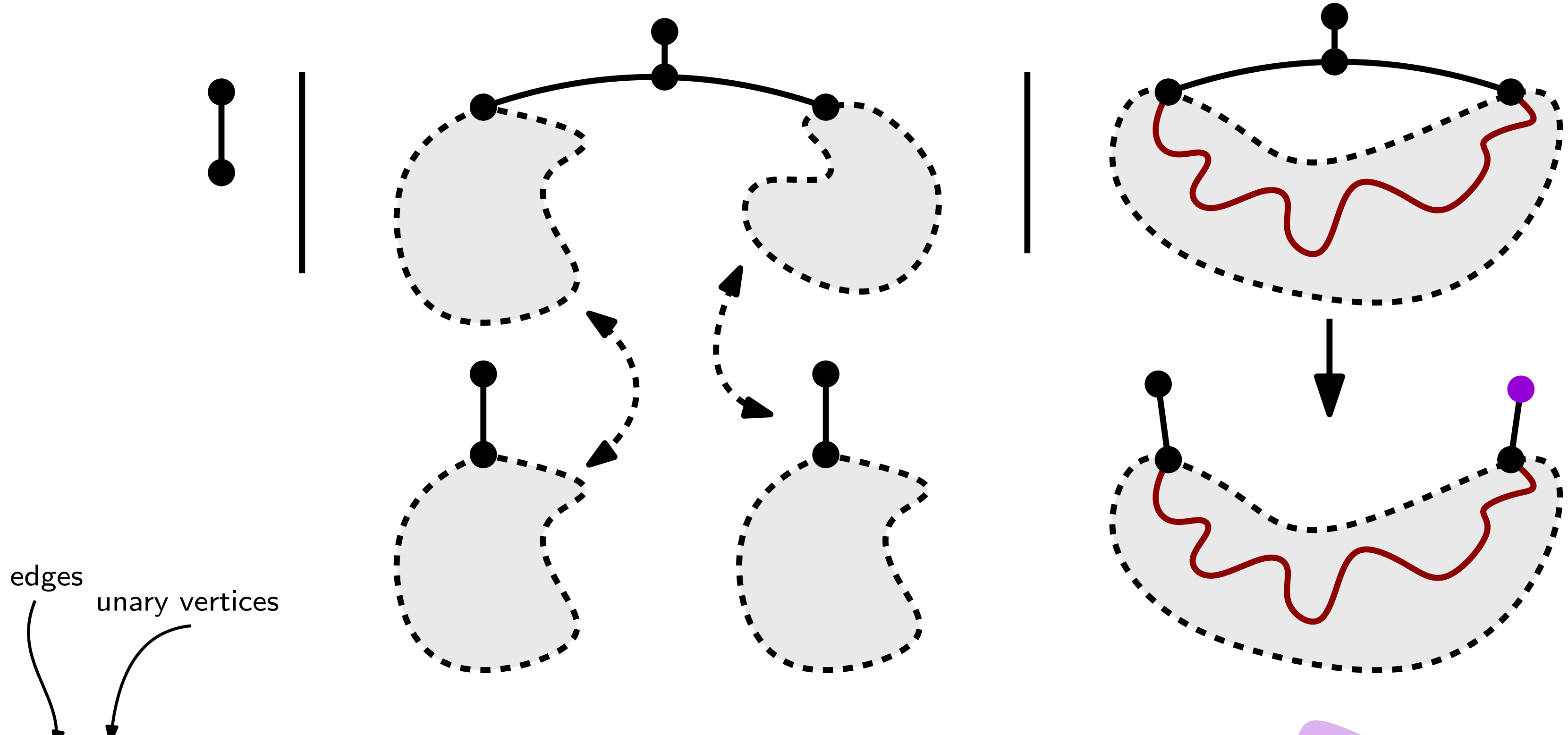

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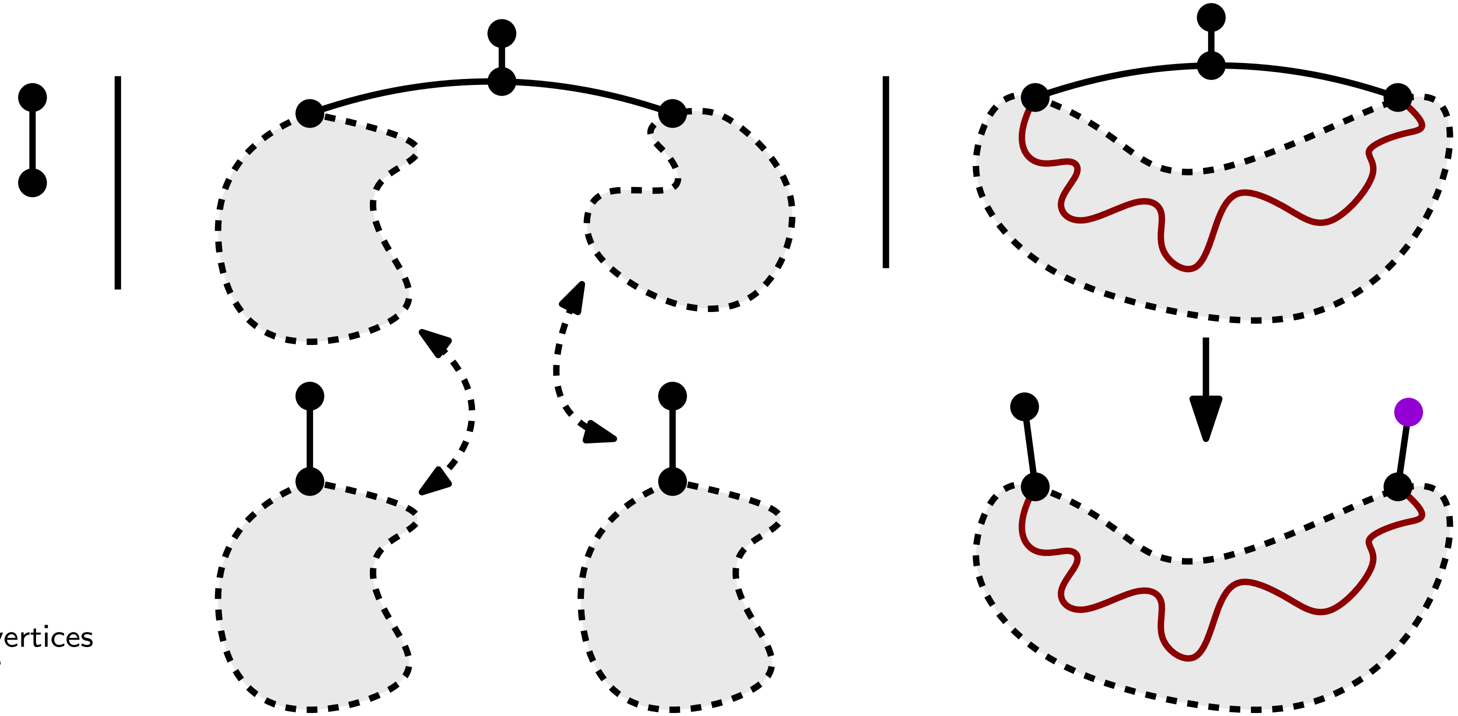

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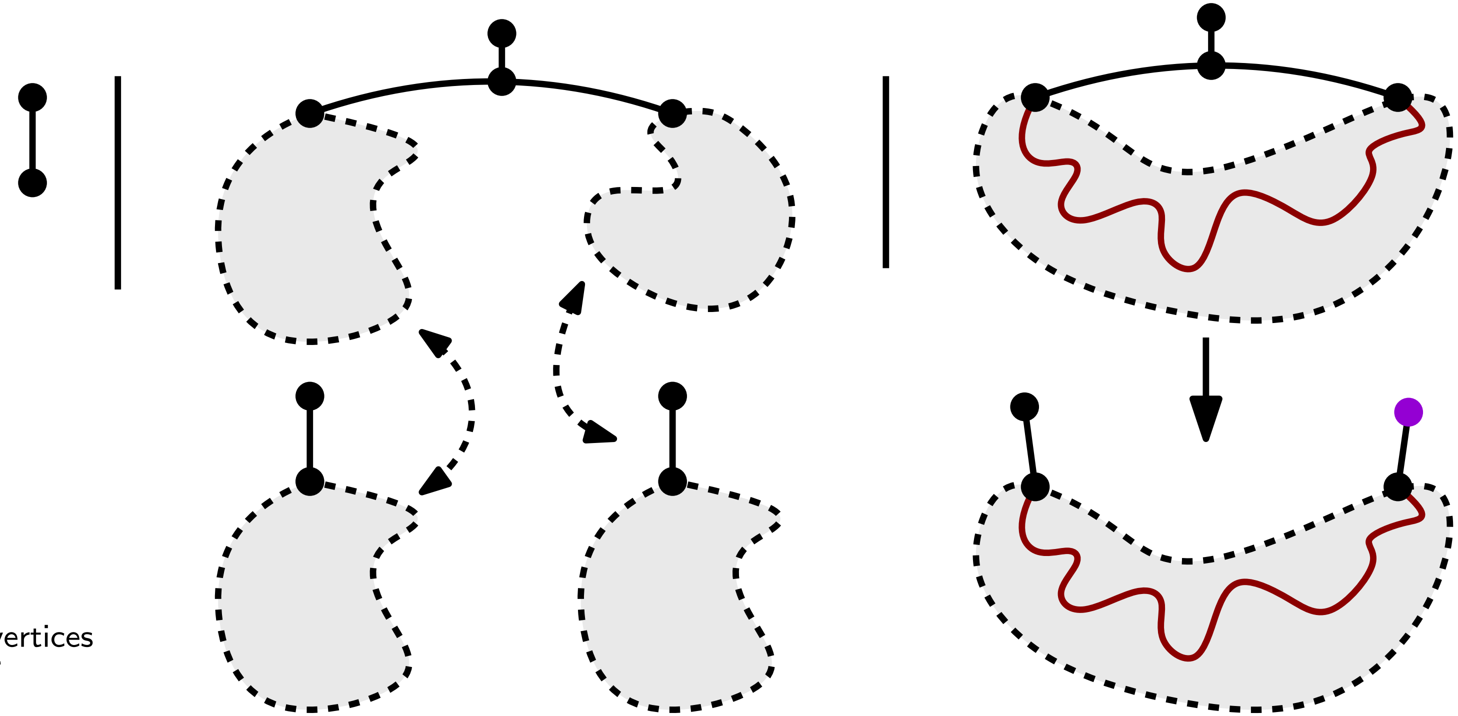

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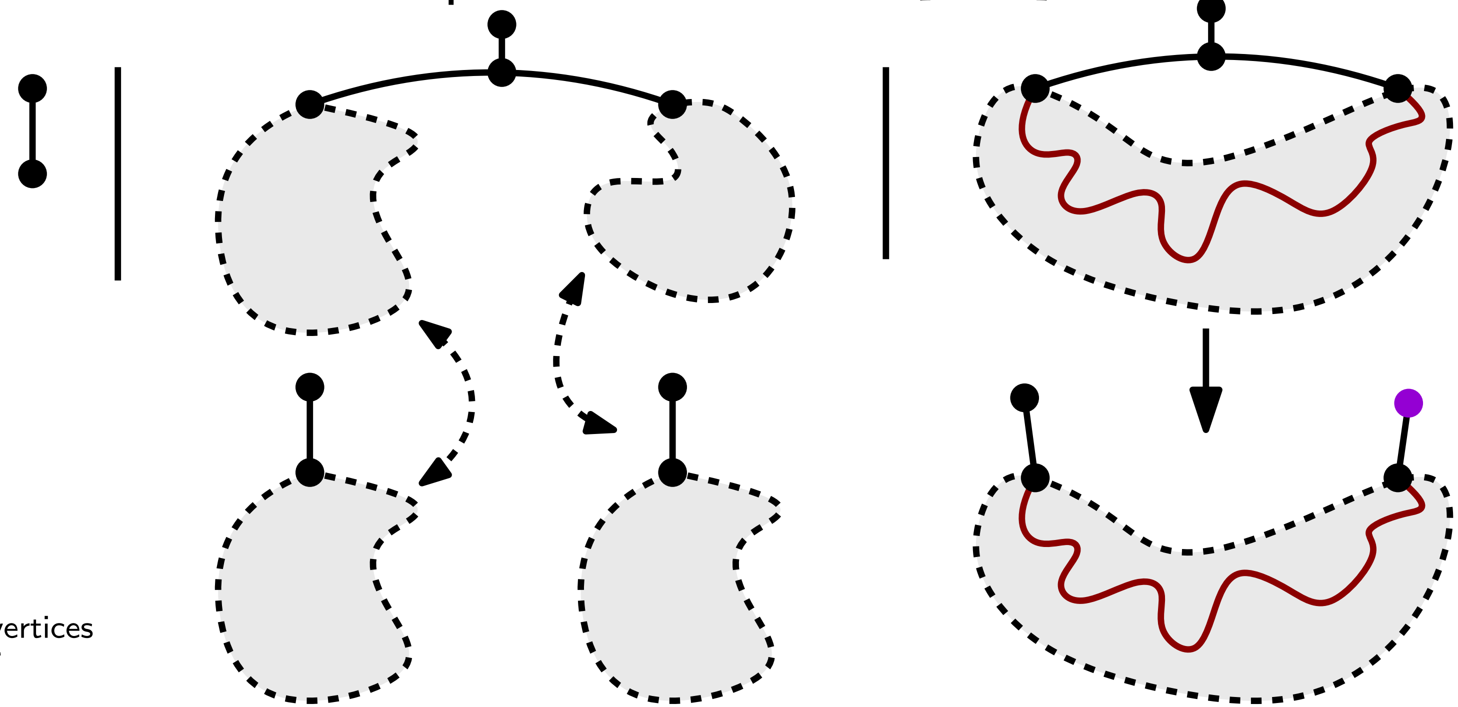
and open linear terms! [Z16]



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subterms free vars

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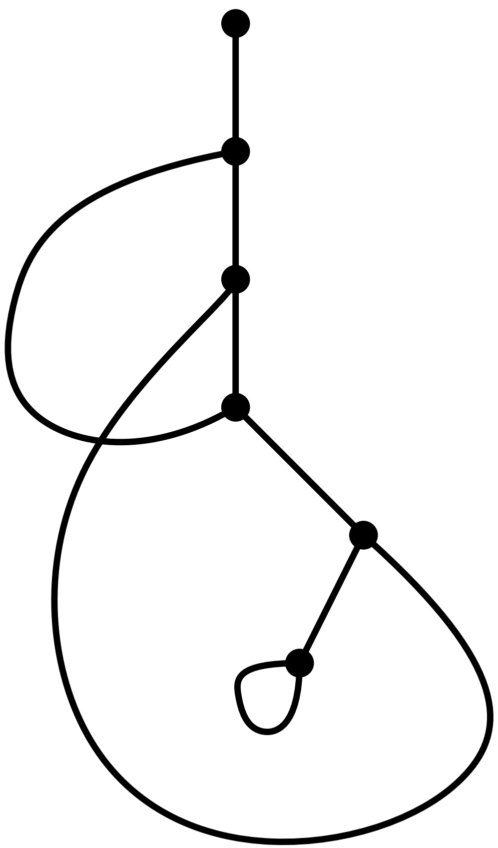
- Rooted (2,3)-maps ↔ closed affine terms

Our results: limit distributions

Closed trivalent maps $\leftrightarrow$ closed linear terms



$$\lambda x.\lambda y.(y\ \lambda w.w)x$$

Our results: limit distributions

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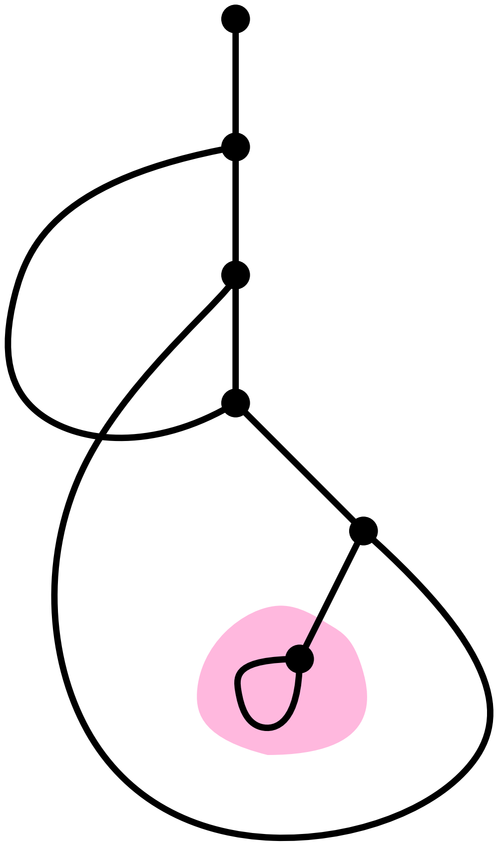
\# loops = # id-subterms



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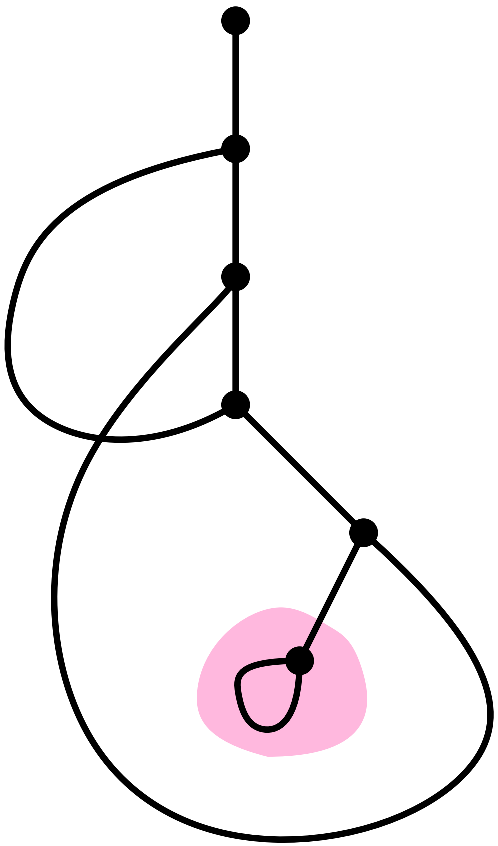

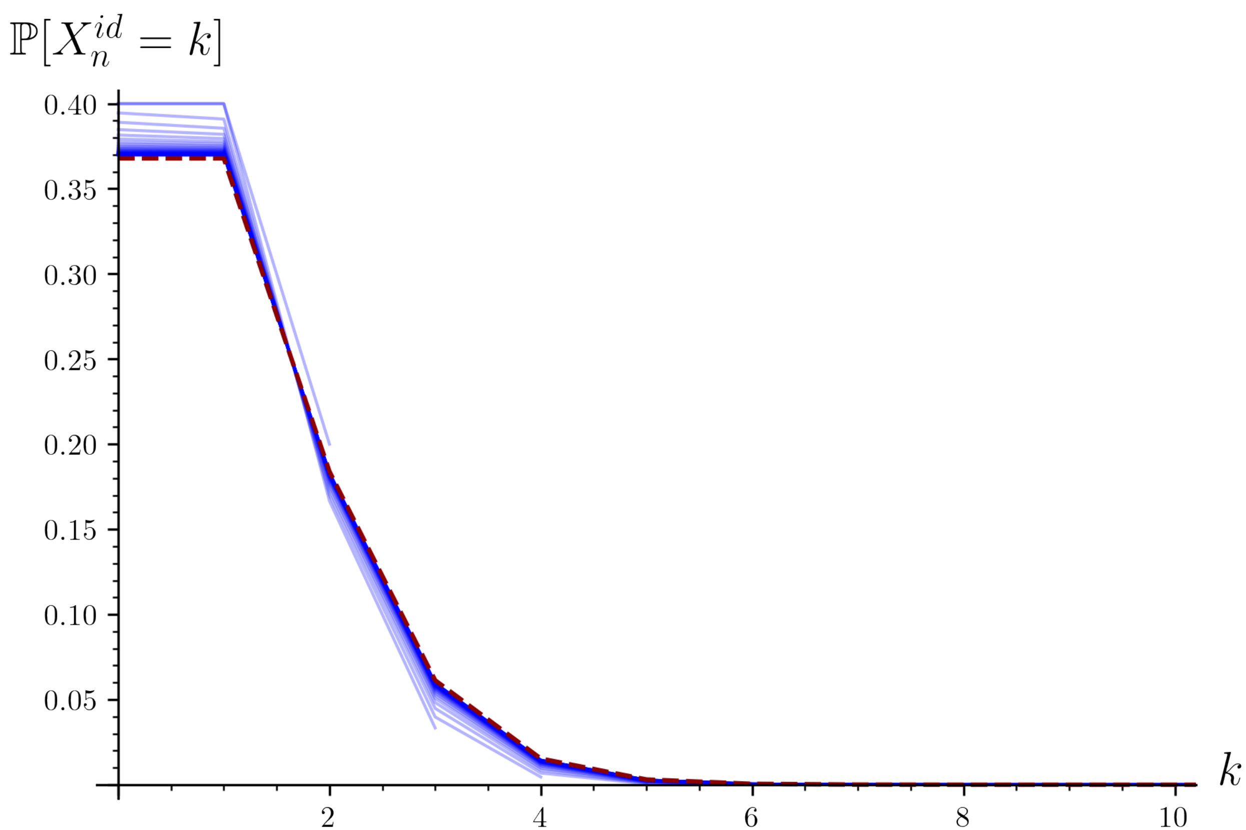

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$$X_n^{id} \xrightarrow{D} \text{Poisson}(1)$$

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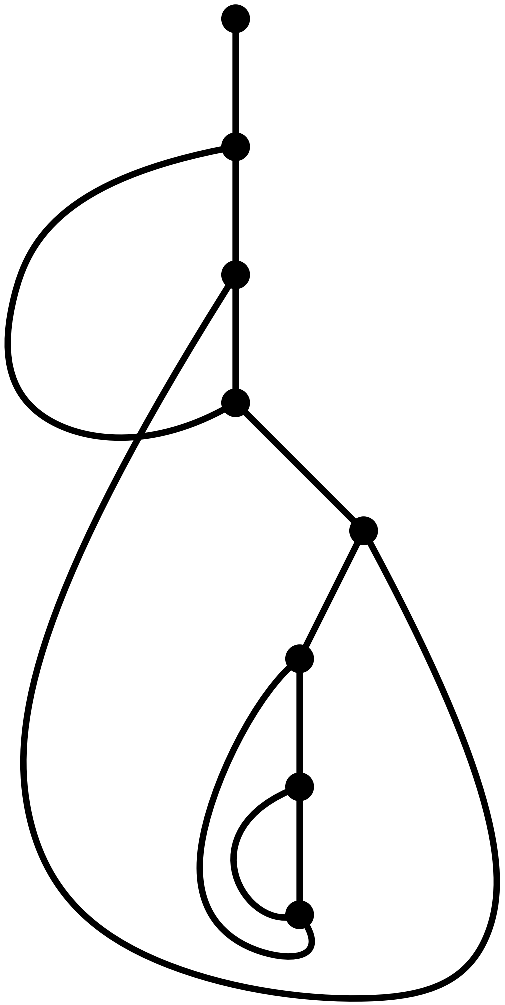

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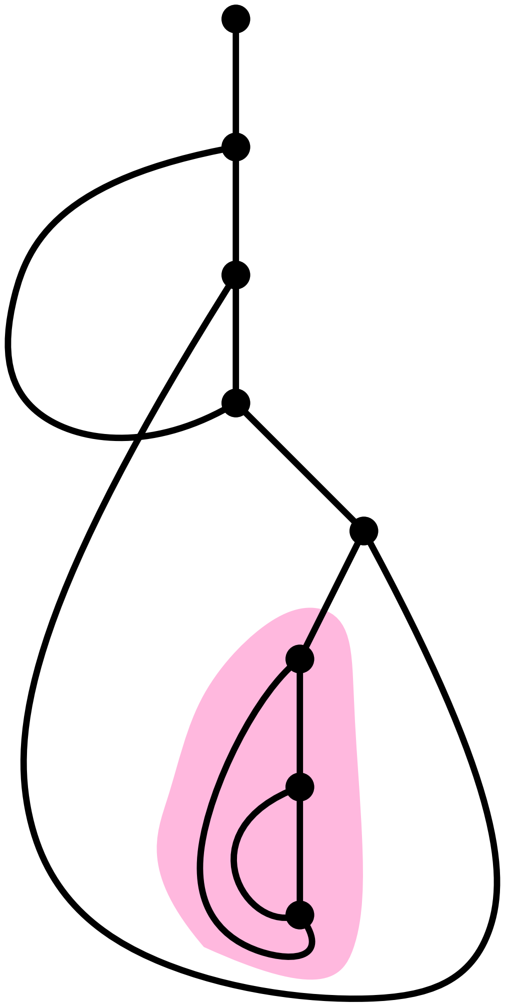

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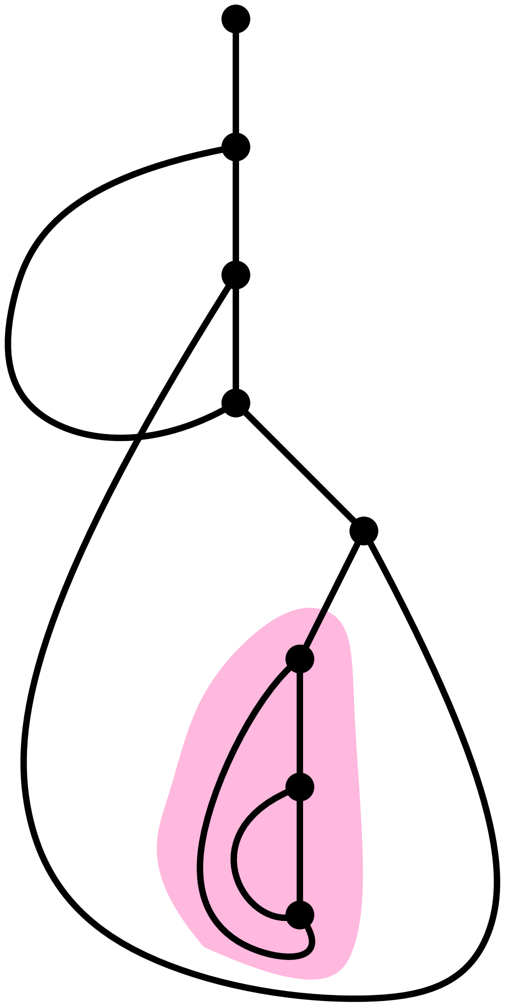

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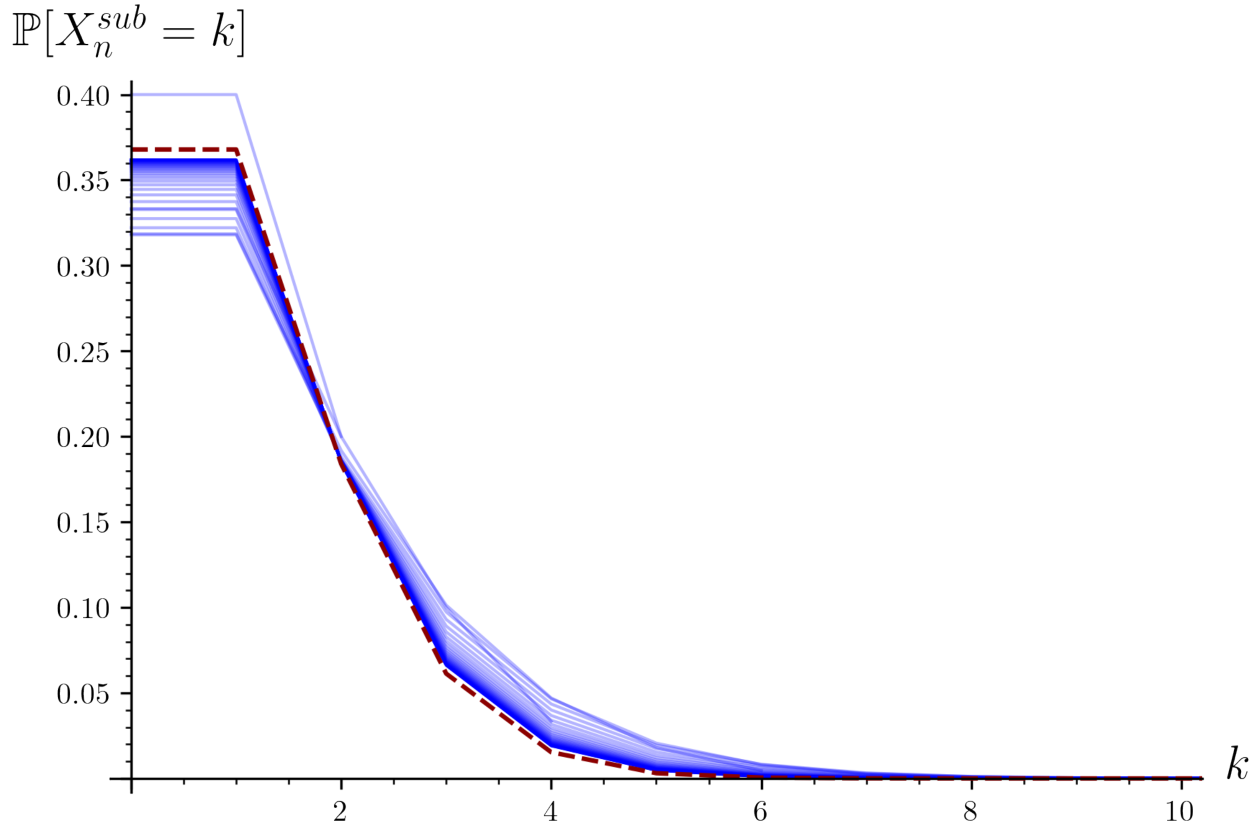

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Closed trivalent maps ↔ closed linear terms

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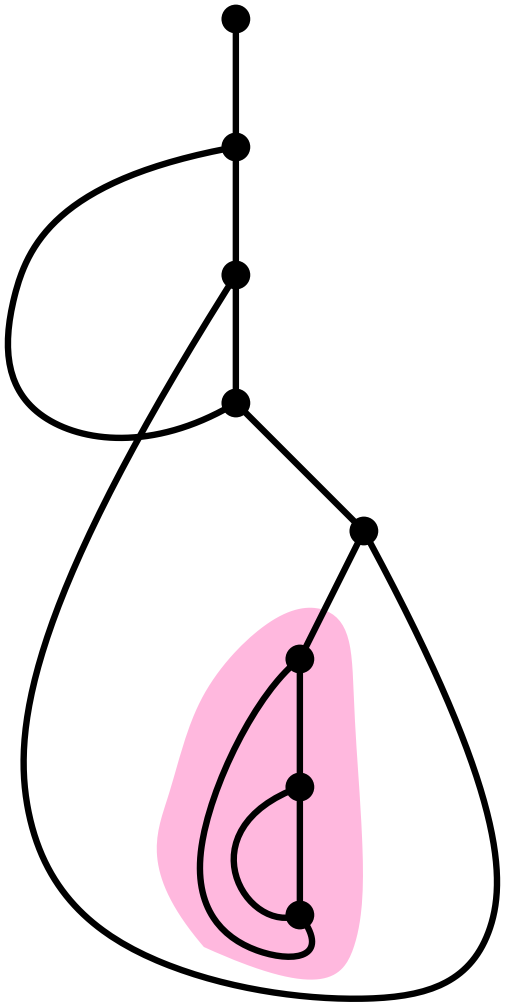
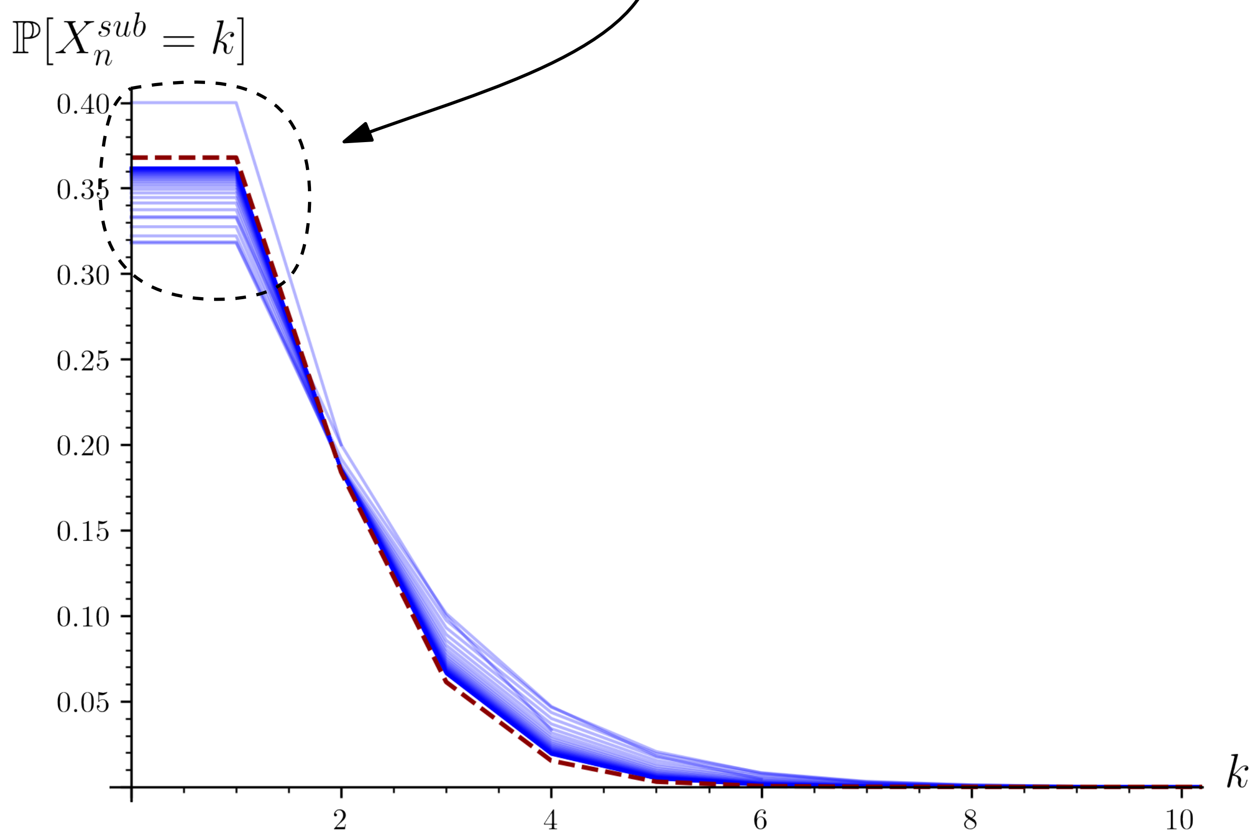


$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$

bad news for remote villages in rooted trivalent maps...

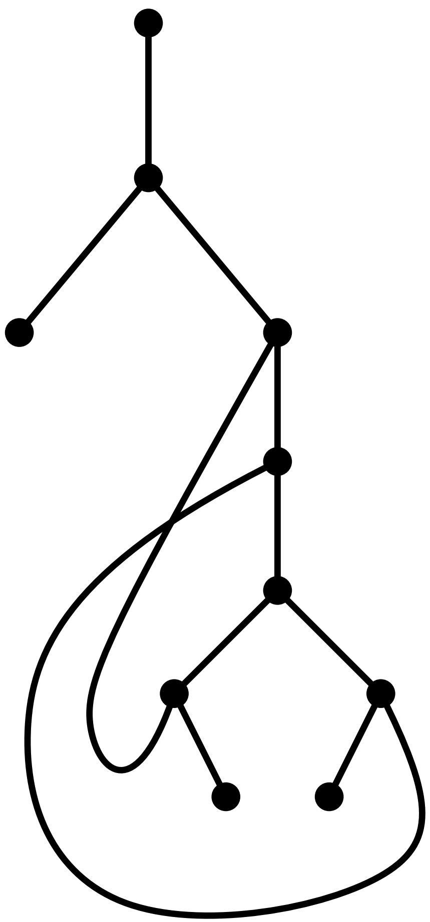
one bridge ↔ no bridge



$X_n^{sub} \xrightarrow{D} \text{Poisson}(1)$

Our results: limit distributions

Open trivalent maps ↔ open linear terms



$(a\ (\lambda x.\lambda y.(y\ b)(c\ x)))$

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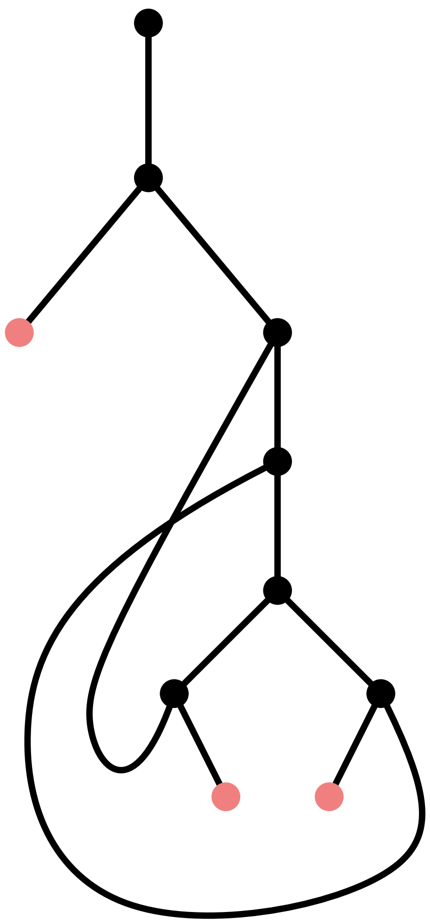
unary vertices = # free vars



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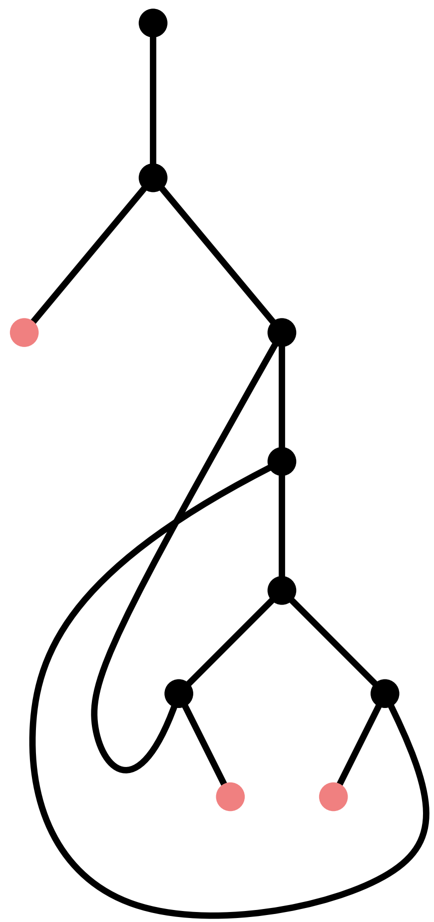

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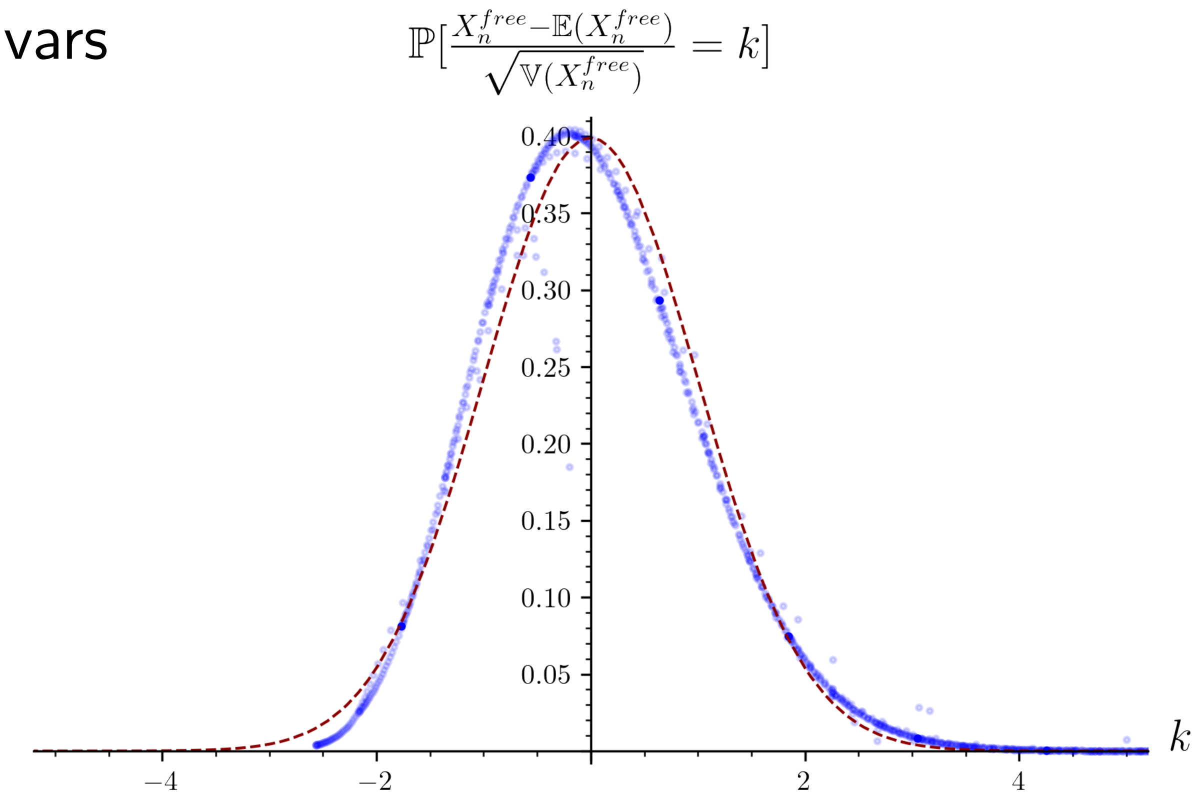
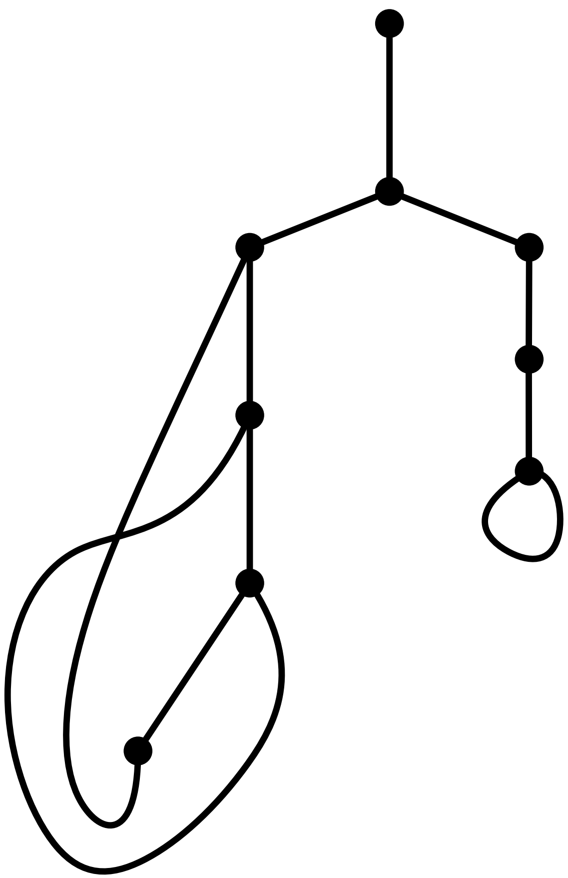


$$\frac{X_n^{free}-\mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1)$$

for $\mu = \sigma^2 = (2n)^{1/3}$

Our results: limit distributions

(2,3)-valent maps ↔ closed affine terms



$(\lambda x.\lambda y.(\lambda z.x)y)(\lambda w.\lambda v.\lambda u.u)$

Our results: limit distributions

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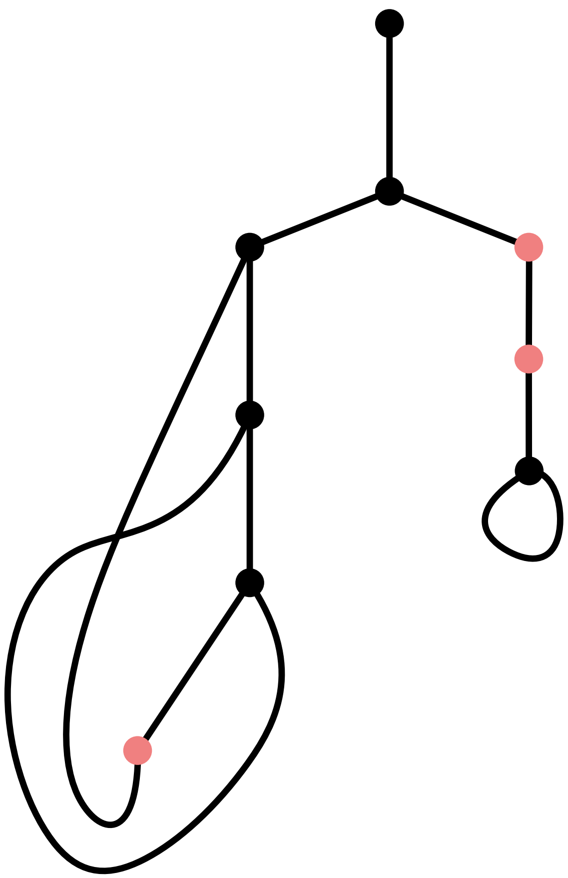

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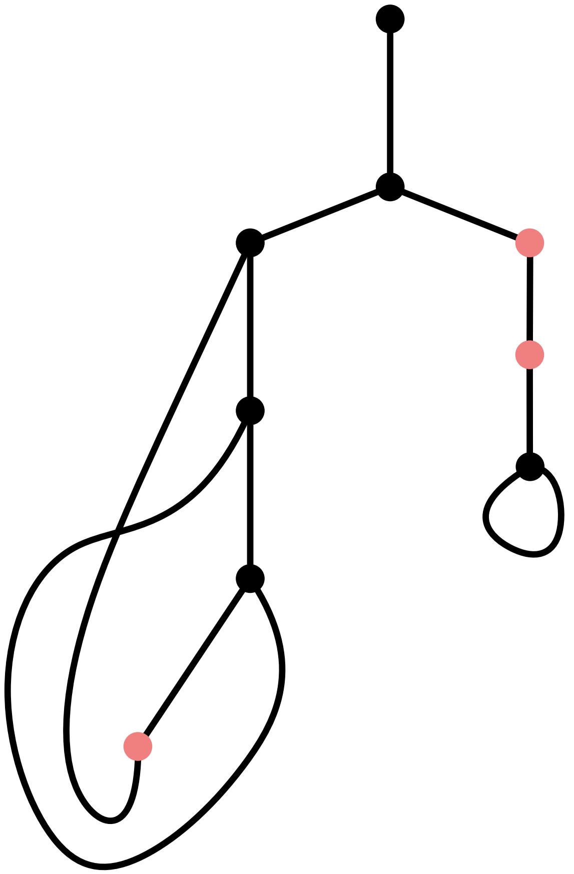

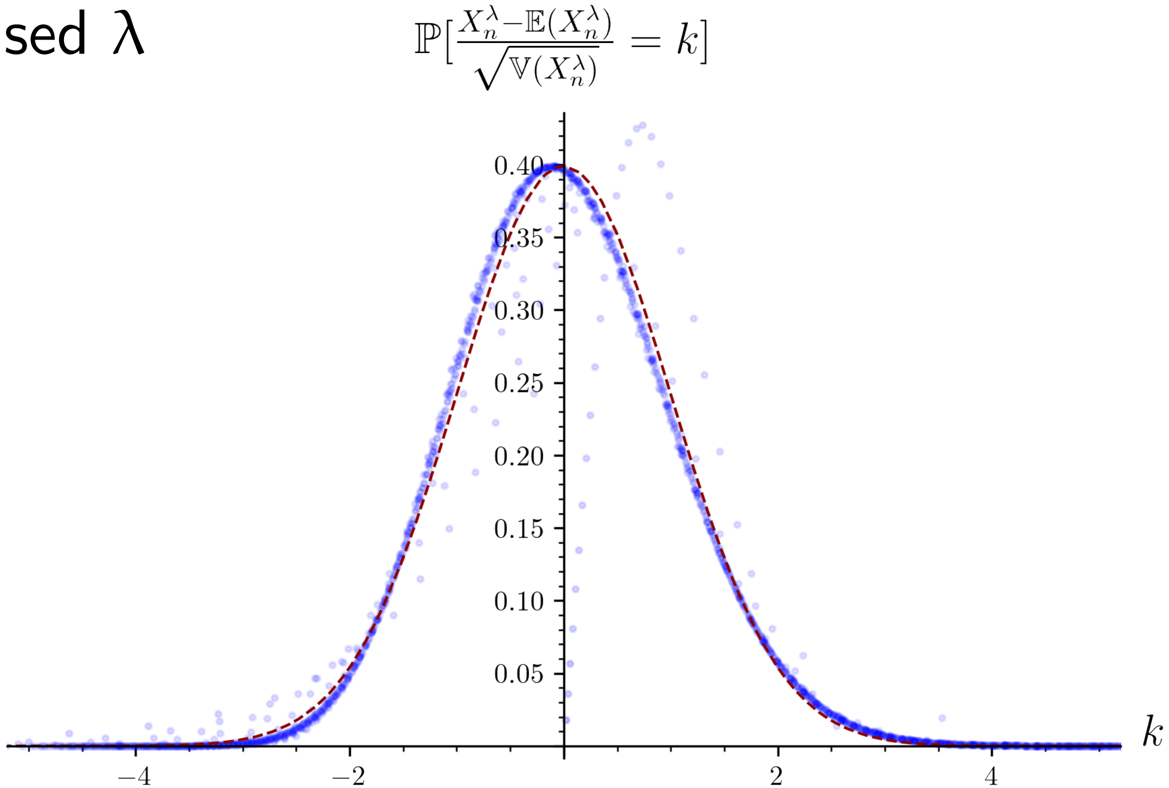

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Our workflow:

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we have a lot of 'em, but only some are tractable!

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- Schema based on compositions (see also [B75,FS93,B18,P19,BKW21]):

$F(z, u, G(z, u))$ inherits the limit law of $G(z, u)$

Proof sketch for loops/id-subterms:

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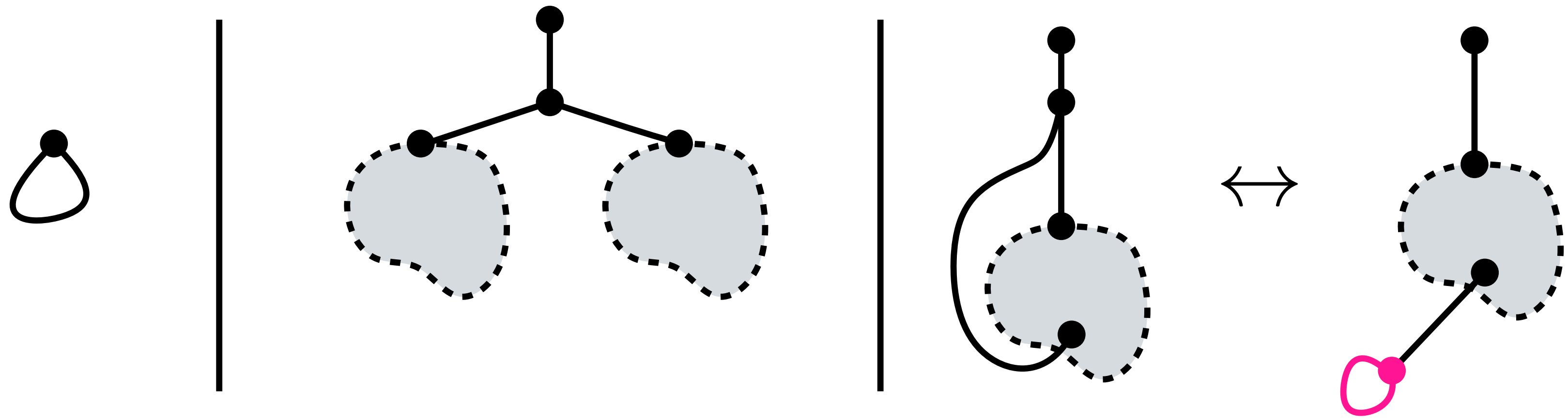
$$T_0^{\mathrm{id}}(z,u) = (u-1)z^2 + zT_0^{\mathrm{id}}(z,u)^2 + \partial_u T_0^{\mathrm{id}}(z,u)$$



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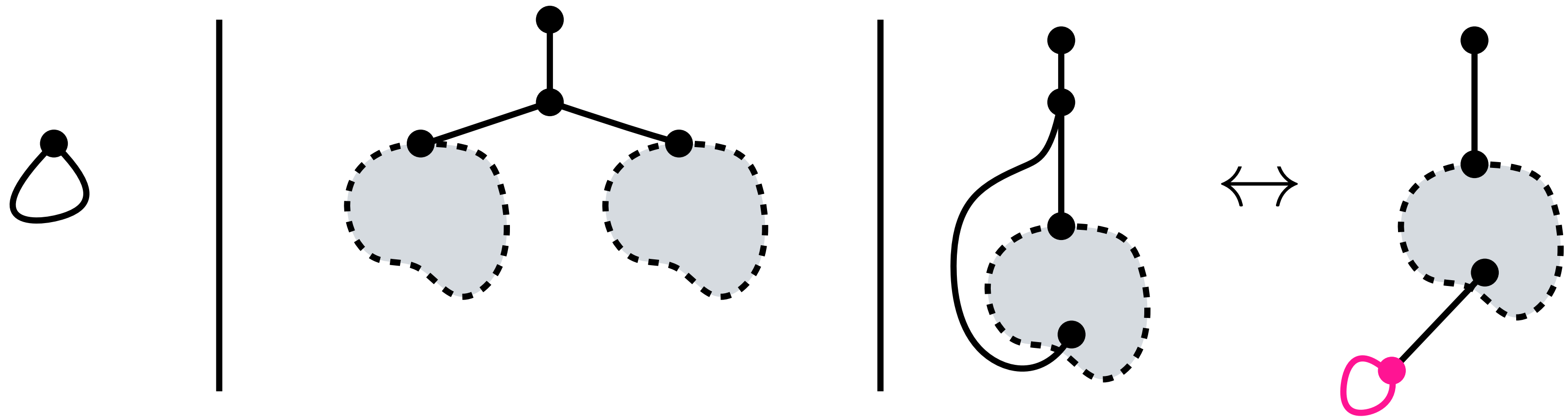

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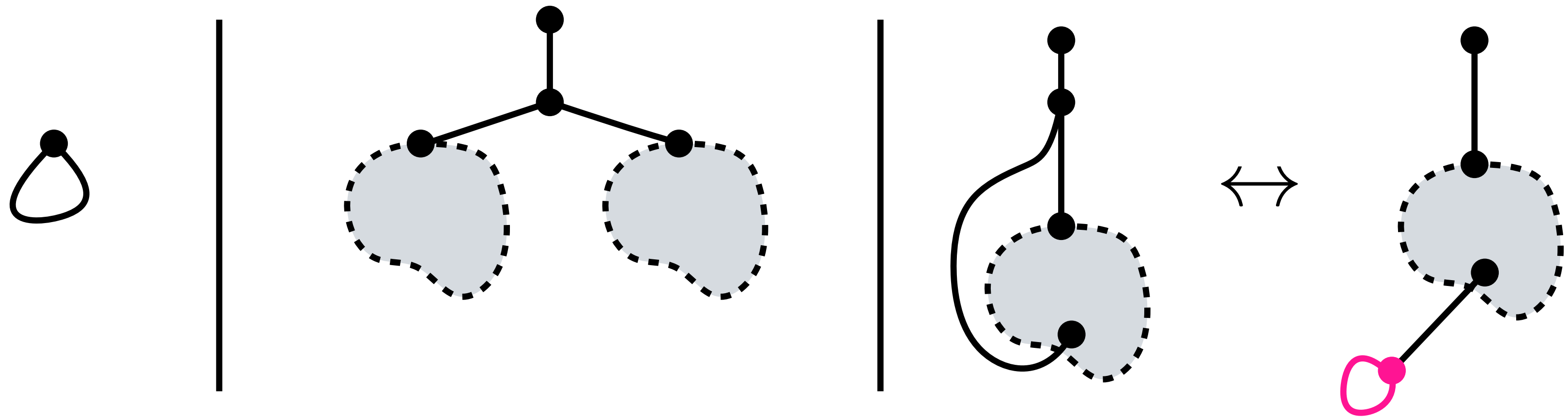

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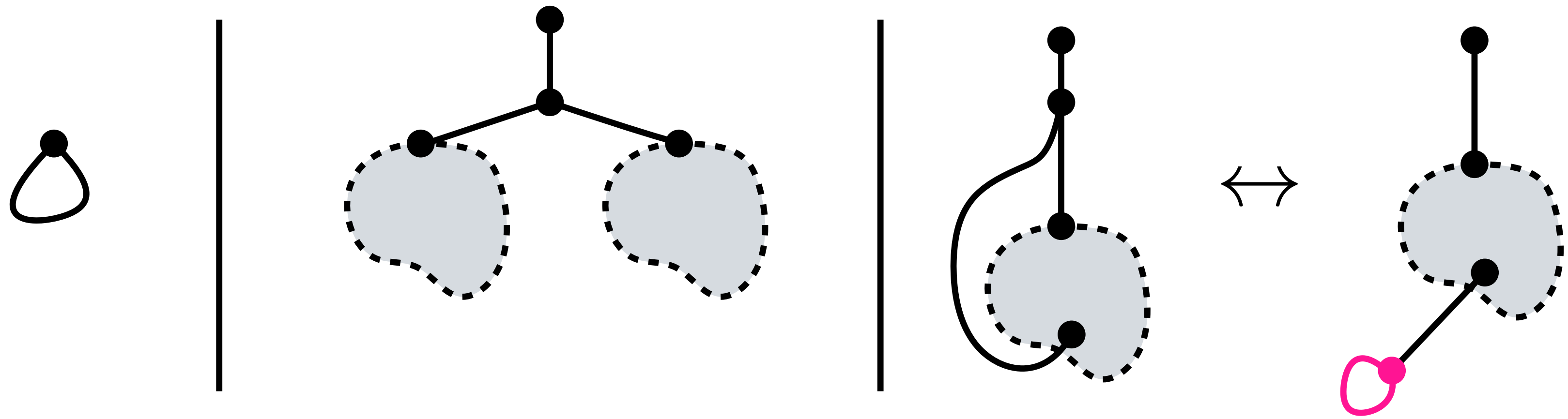

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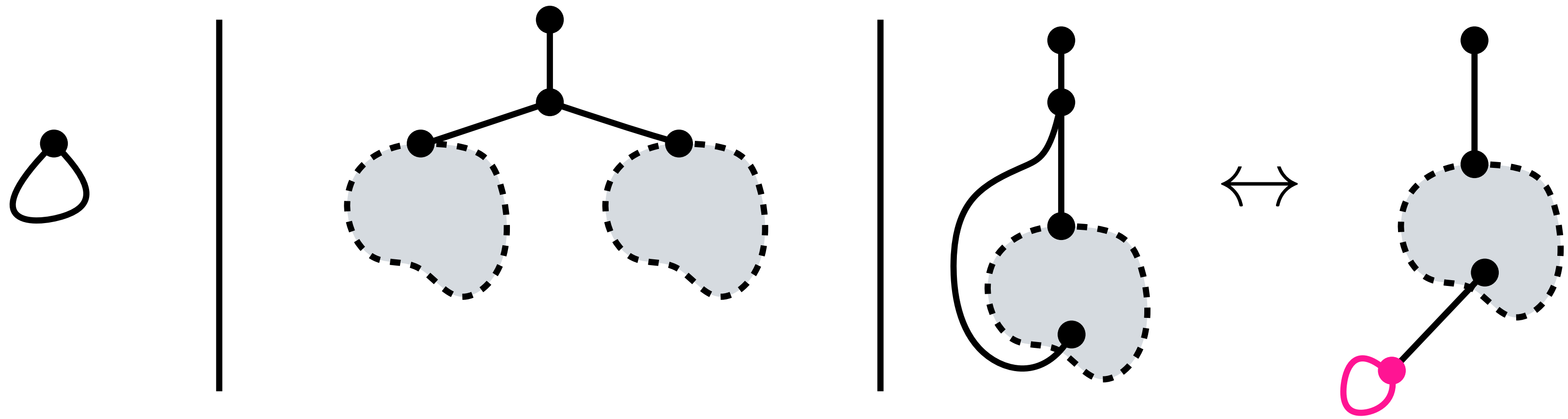

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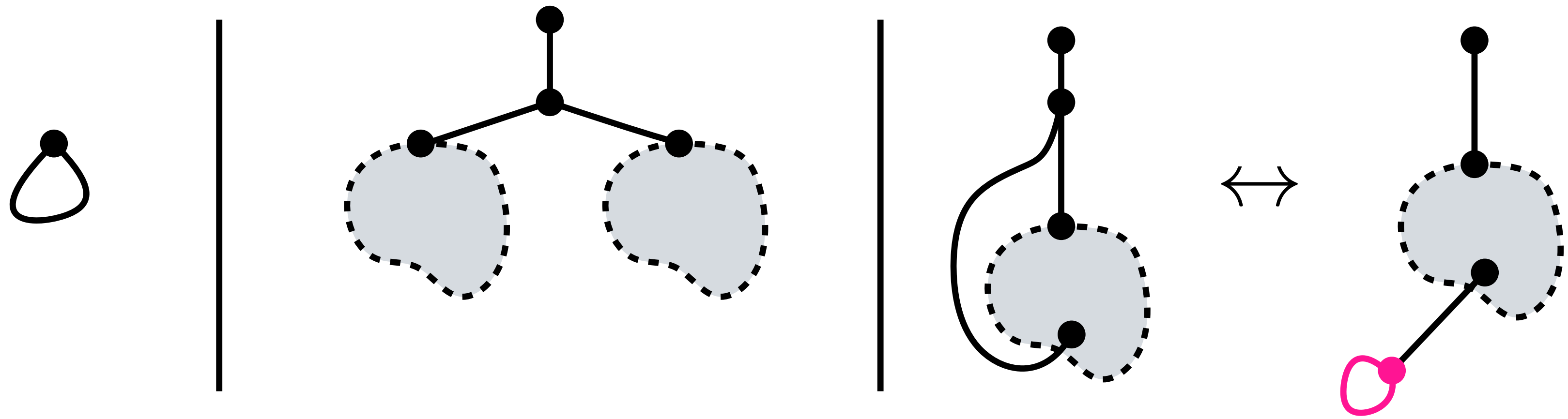

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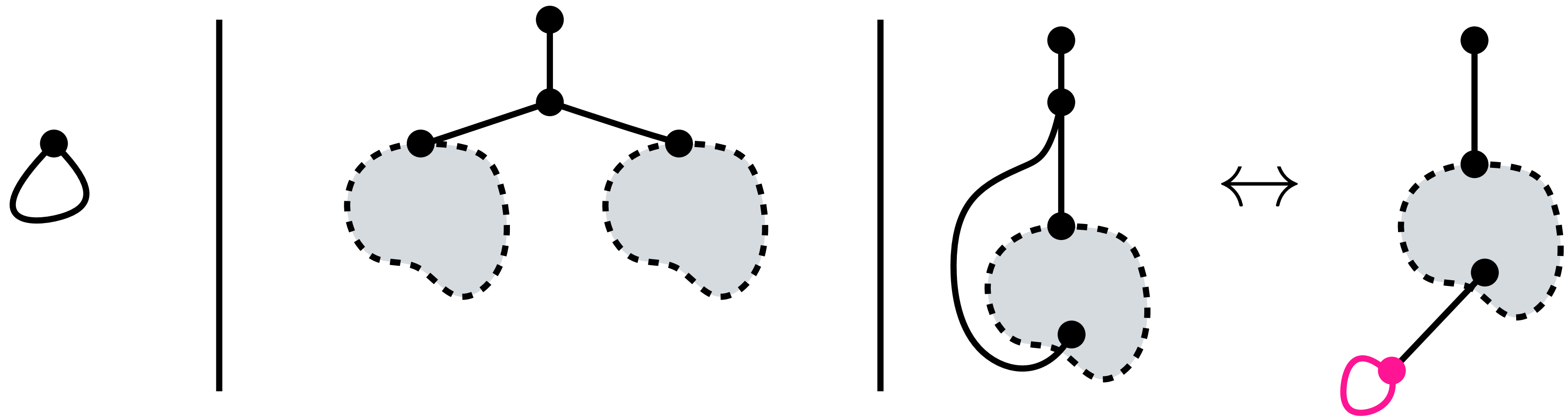

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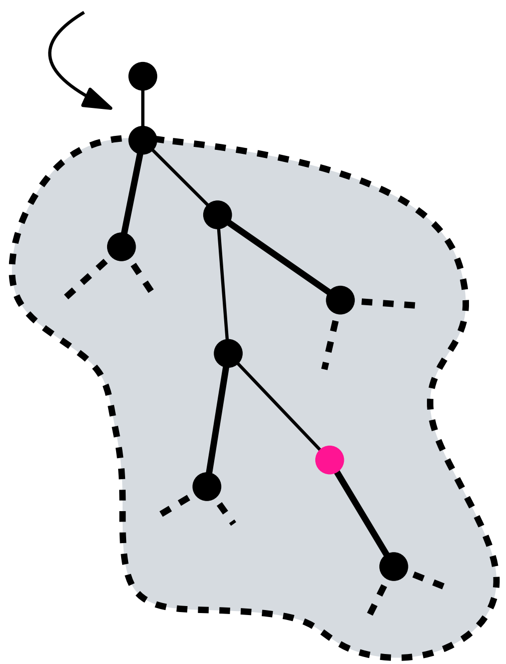
Schema then yields Poisson(1) limit law

Proof sketch for bridges/closed subterms:



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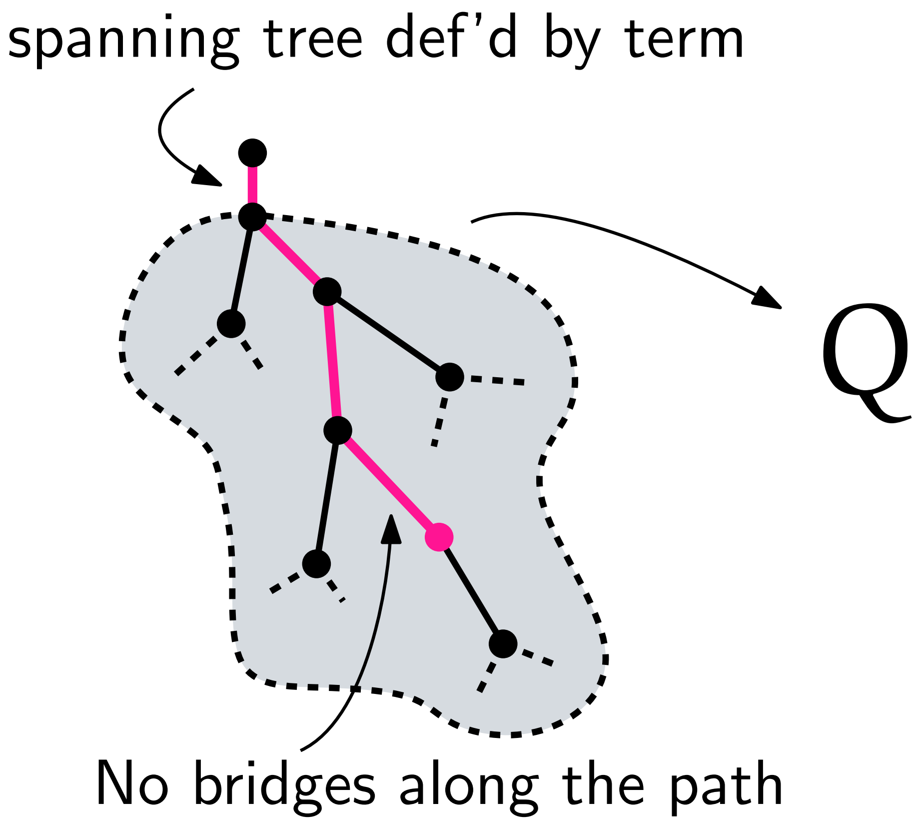

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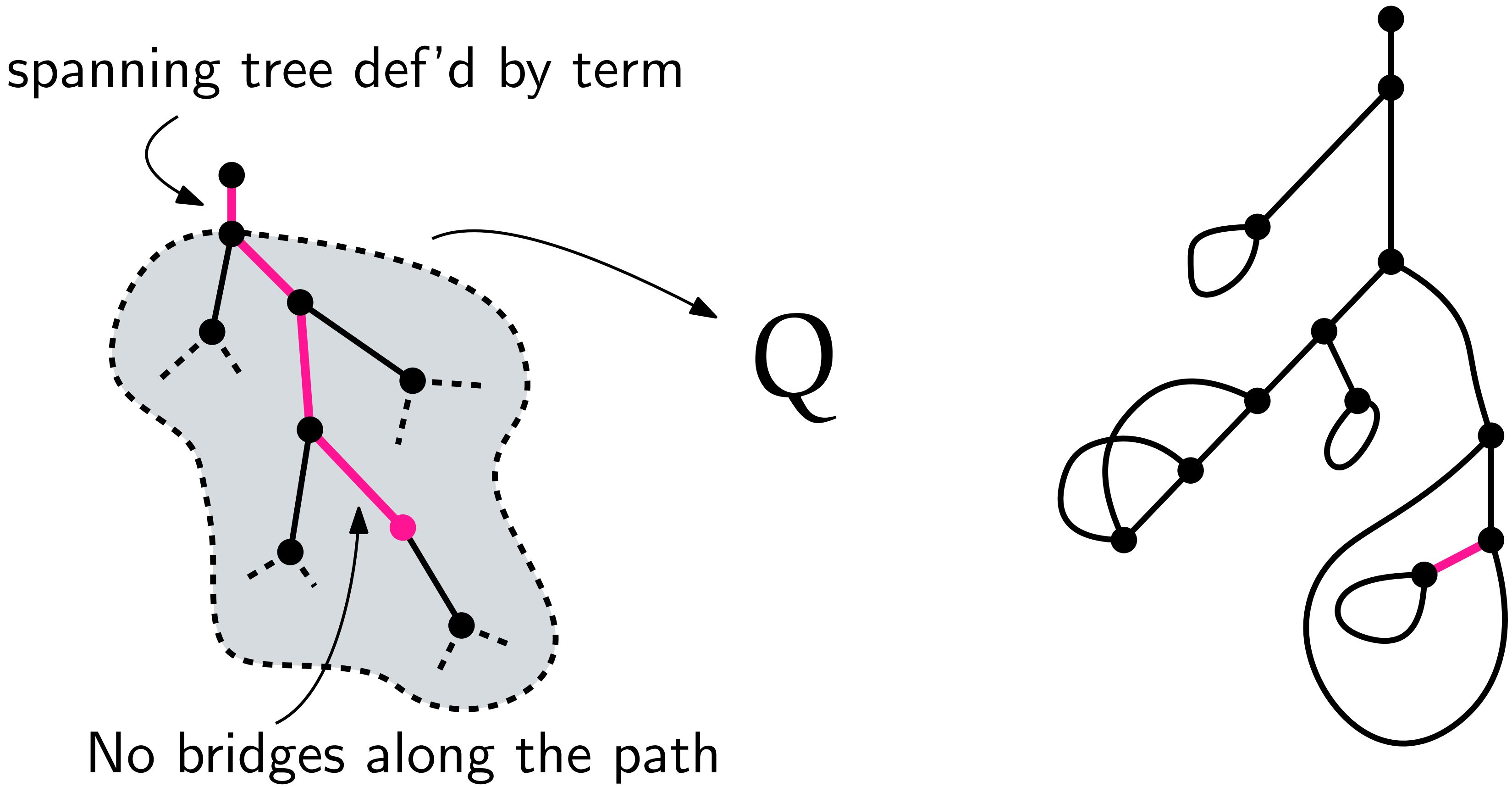

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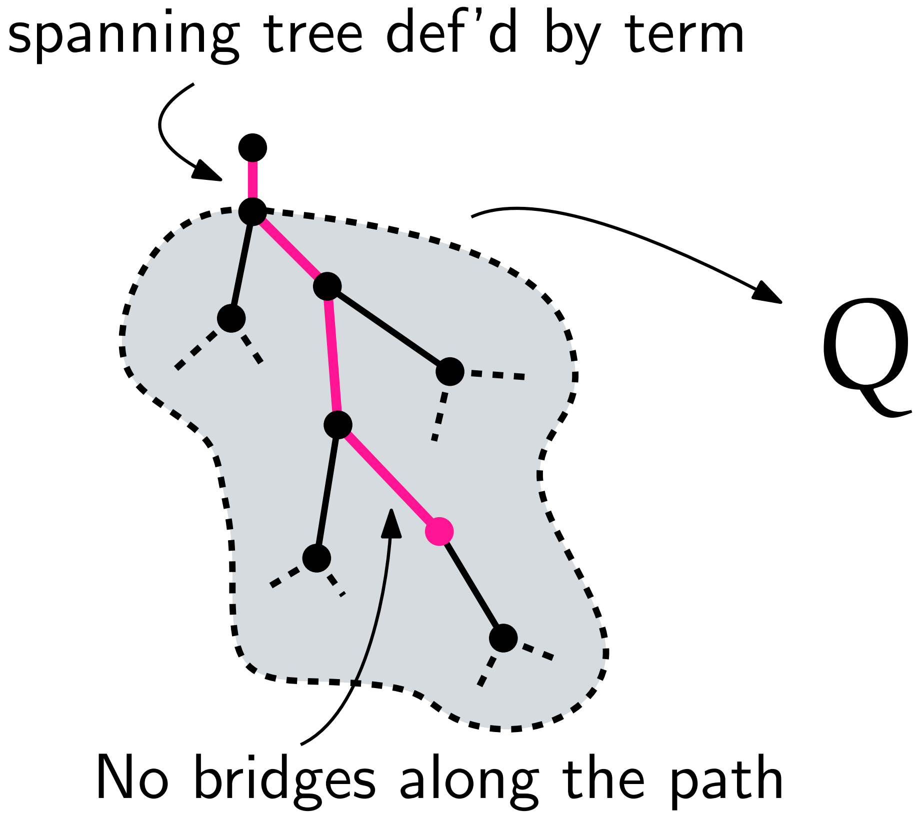

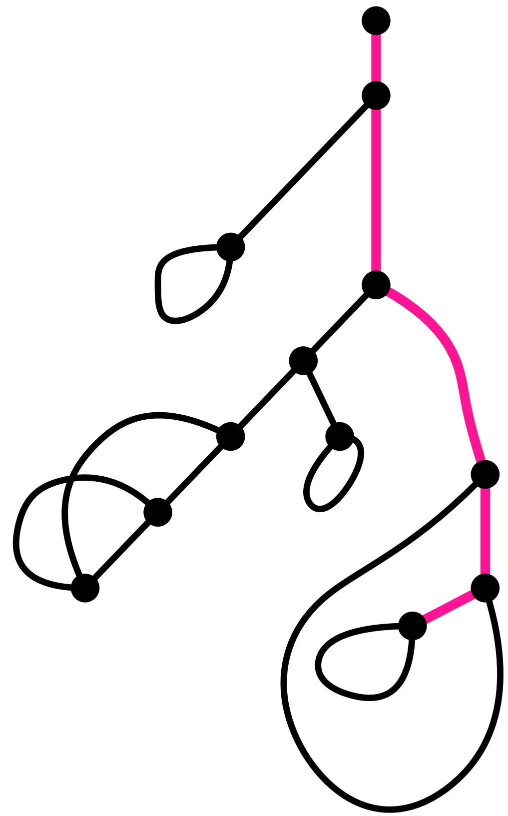

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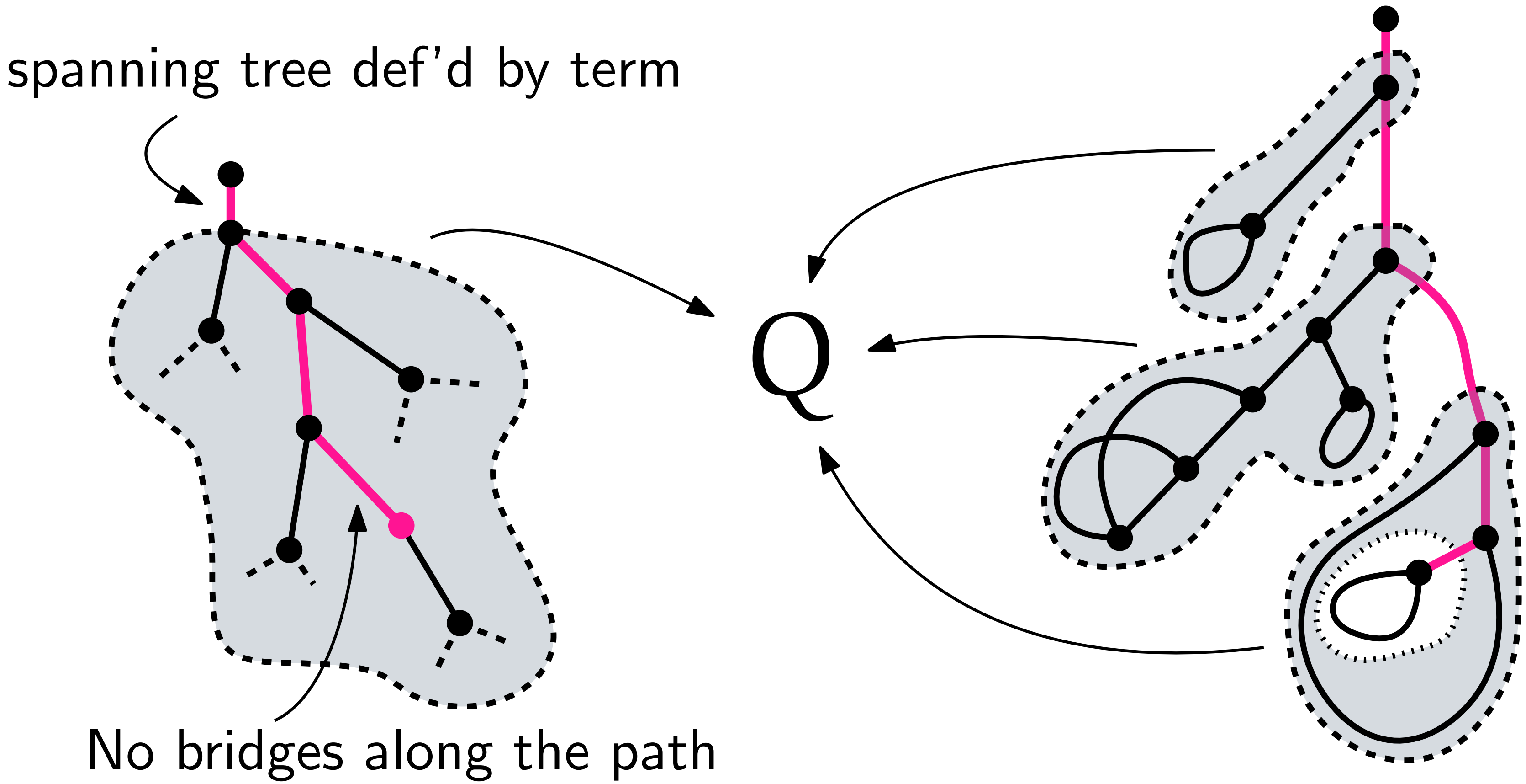

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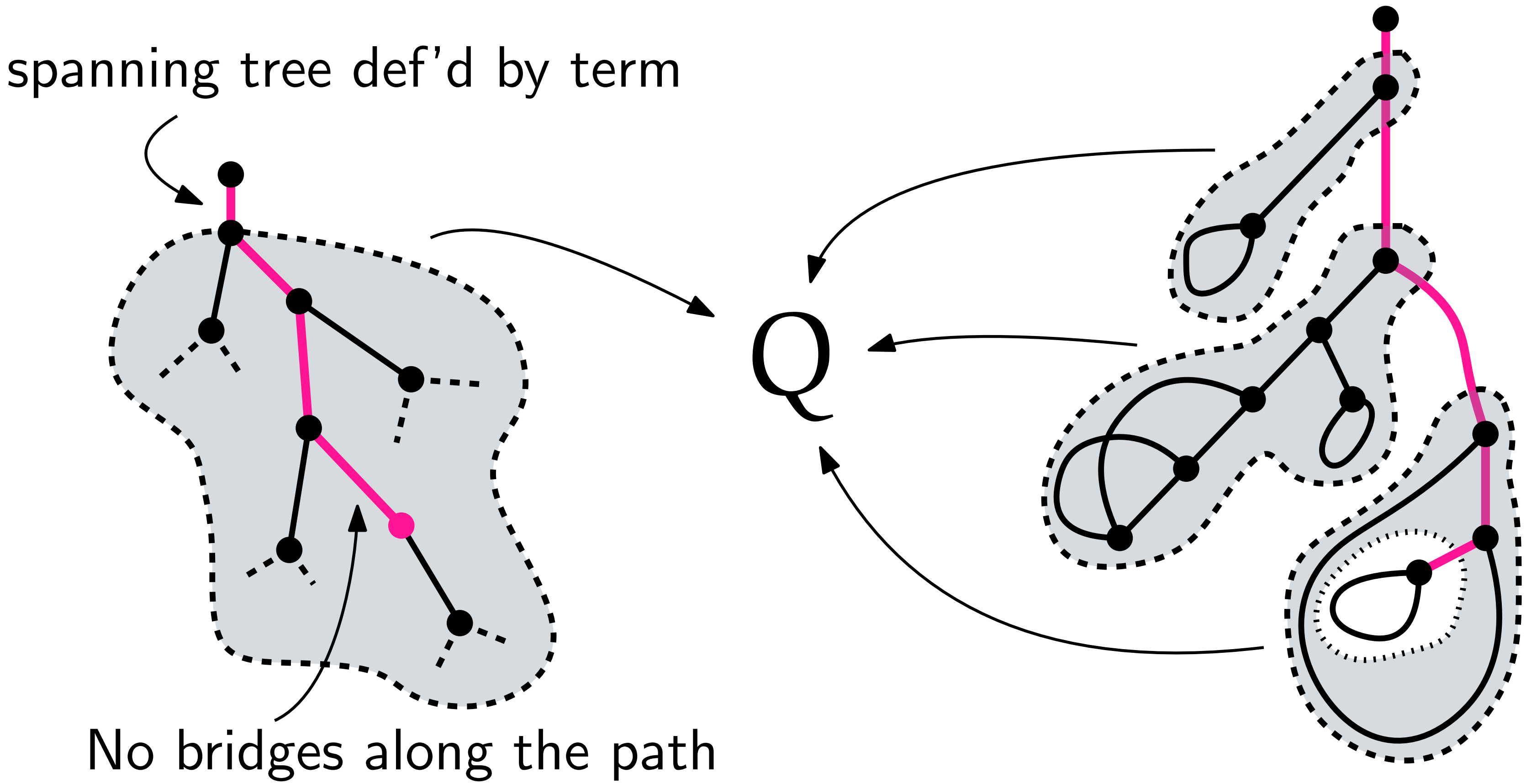

$$\frac{\partial}{\partial v} T_0^{sub}(z, v) = -\frac{v^2 z T_0^{sub}(z,v)^3 + z^2 T_0^{sub}(z,v) - T_0^{sub}(z,v)^2}{(v^3 - v^2) z T_0^{sub}(z,v)^2 + v z^2 - (v-1) T_0^{sub}(z,v)}$$

May be pumped using our schema

Proof sketch for vertices of given degree:

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Specifications based on exponential Hadamard products

$$\mathrm{OT}(z,u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z}\left(\ln\left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$$

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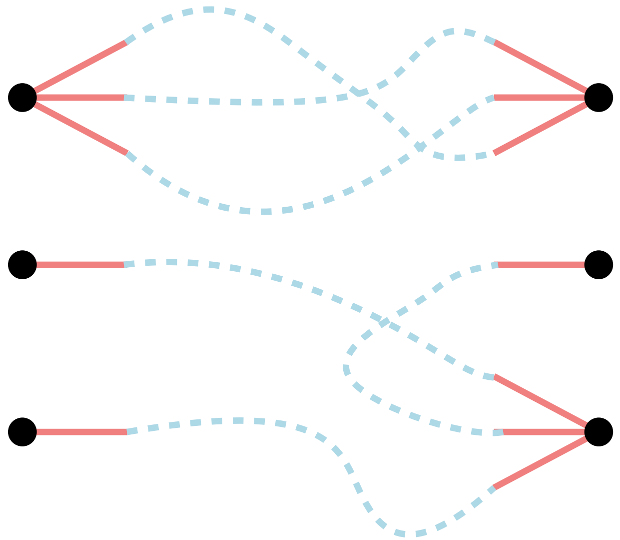

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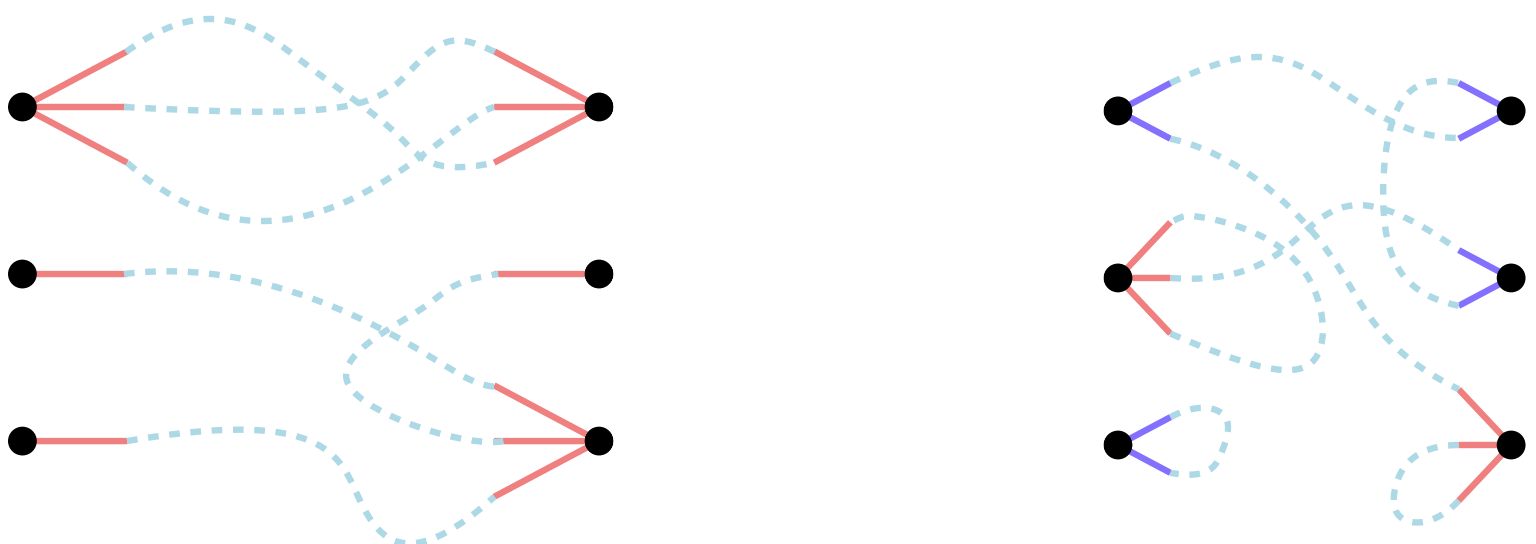

(2,3)-valent maps

$$TT(z,u) = z\frac{\partial}{\partial z}\left(\ln\left(\exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + \frac{uz^2}{2}\right)\right)\right)$$

$$A(z,u) = \frac{z^2 + z^2 TT(z^{\frac{1}{2}}, u)}{1-z}$$

closed affine terms

Compositions for fast-growing series:

$$F(z, u, G(z, u))$$

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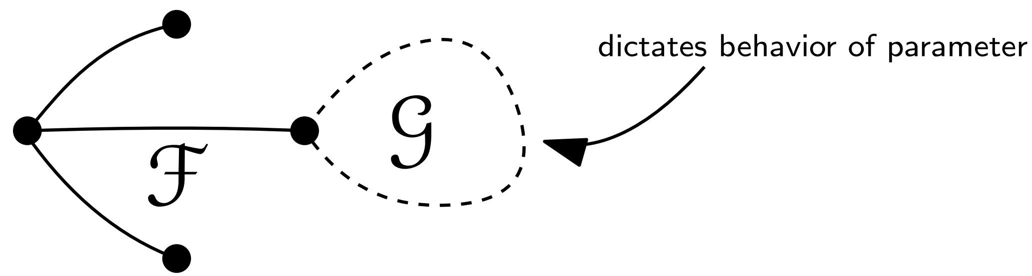
"To build a big $\mathcal{F}(\mathcal{G})$ structure, pick a small $\mathcal{F}$ one and replace one of its atoms with a big $\mathcal{G}$-structure"



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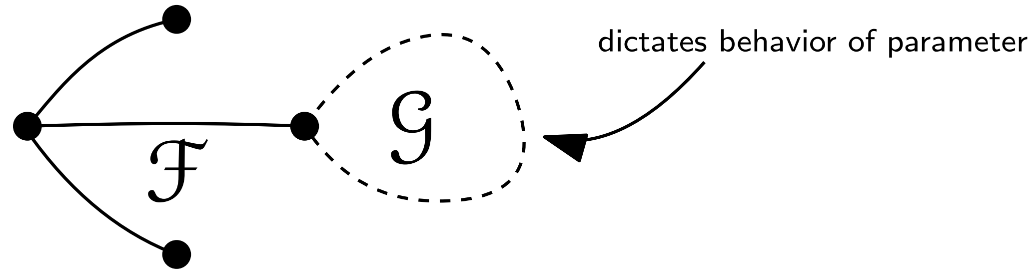

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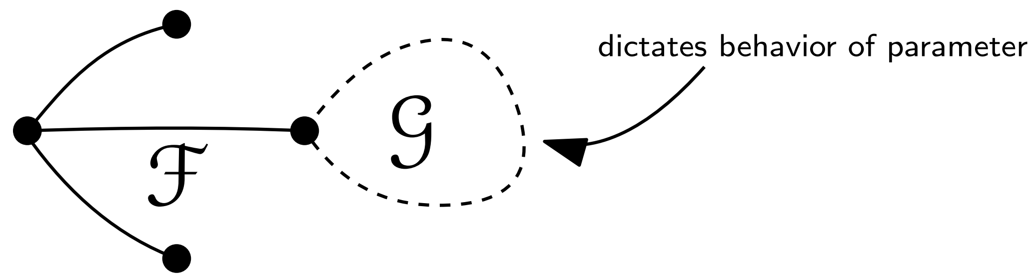

If F is the logarithm:

Asymptotically, almost all not-necessarily-connected $\mathcal{G}$-structures are connected, so the distribution of params. is the same for connected and not-necessarily-so structures!

Proof sketch for bridges/closed subterms (contd.) :

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Ammenable to saddle-point analysis!

Both yield Gaussian limit laws

Proof sketch for bridges/closed subterms (contd.) :

rooted → $\frac{\partial}{\partial z}$; connected → $\ln$

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Ammenable to saddle-point analysis!

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Use schema for compositions to show that the results carry over!

Whats next?

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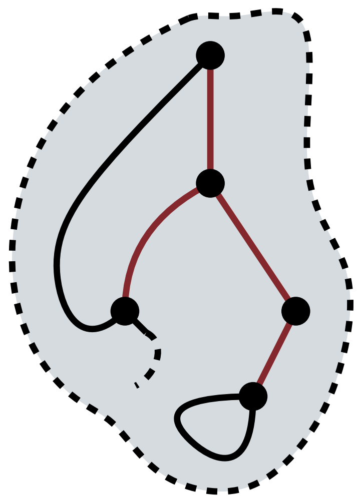
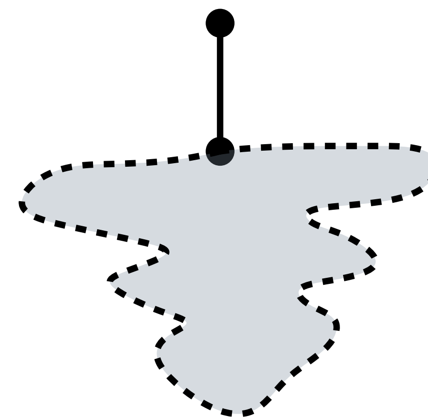
- More parameters:



Mean path length



Profile

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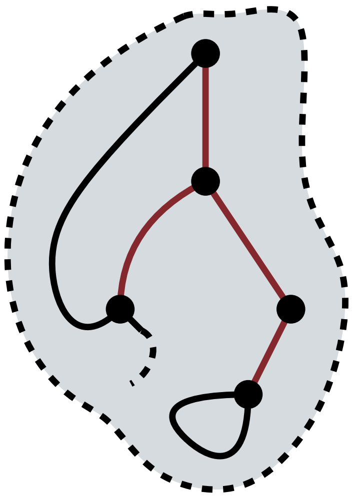

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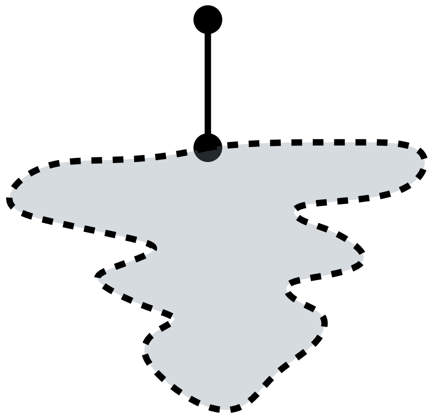

Profile

- More map/term families: planar, bridgeless...

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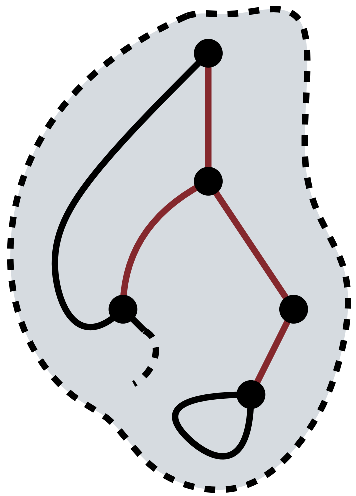

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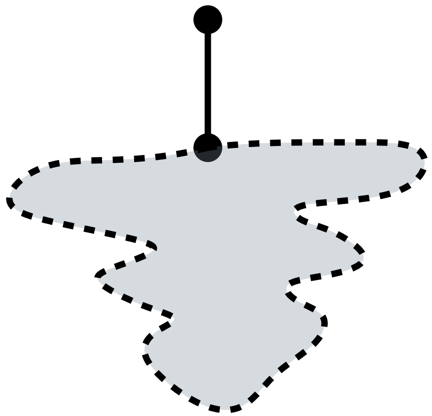

Profile

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Thank you!

Bibliography

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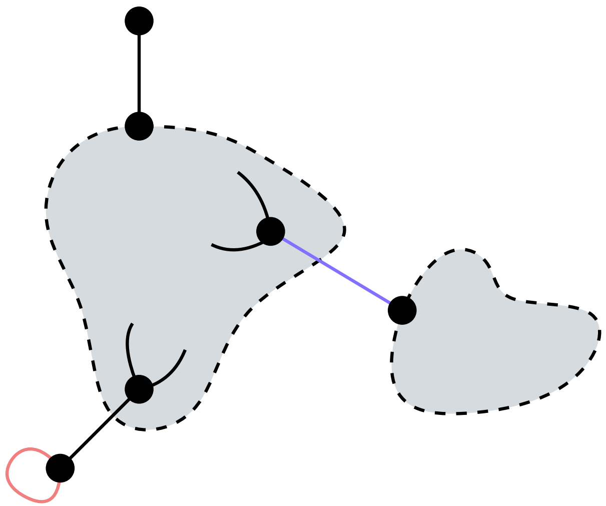
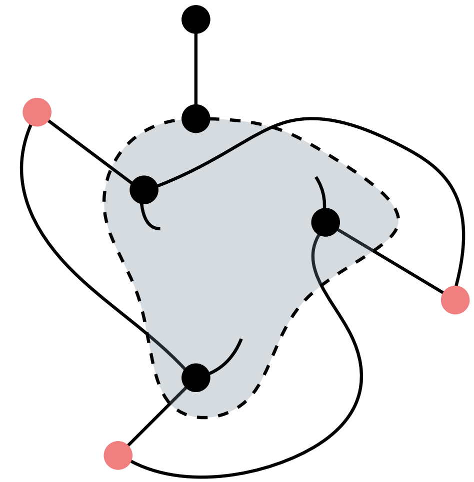
Our results: limit distributions

Trivalent maps ↔ closed linear terms



loops = #id-subterms
bridges = # closed subt.
} Poisson(1)

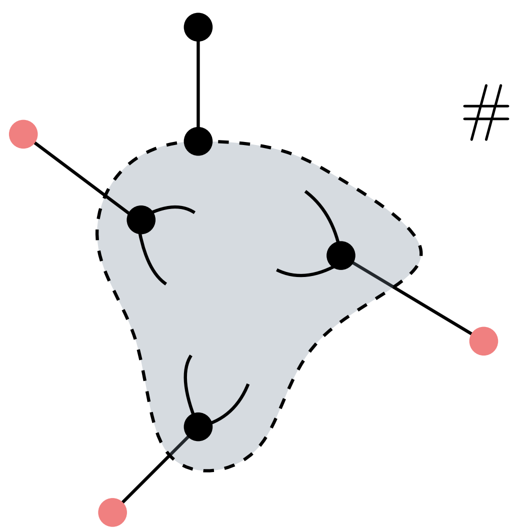
(2,3)-maps ↔ closed affine terms



unary vertices = # free vars

$\mathcal{N}(mu, \sigma^2)$ with $\mu = \sigma^2 = (2n)^{2/3}$

(1,3)-maps ↔ open linear terms



unary vertices = # free vars

$\mathcal{N}(\mu, \sigma^2)$ with

$\mu = \sigma^2 = (2n)^{1/3}$