Bisimulation between hom sets and logics without counting

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The setting

Let σ be a finite relational vocabulary, we can define a category of σ-structures Σ:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R \in \sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- Embeddings $f : \mathcal{A} \rightarrowtail \mathcal{B}$ are injective morphisms that reflect relations:

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Leftarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- Set of homomorphisms $\mathrm{Hom}_{\Sigma}(\mathcal{A}, \mathcal{B})$

Subcategory of finite σ-structures Σ_f

Lovász-type results

$\mathcal{A}, \mathcal{B}$ be finite σ-structures. $\mathcal{V}^k(\#)$ and $QR_n(\#)$ are k-variable logic and logic up to quantifier rank $\leq n$ with counting quantifiers, i.e. $\exists_{\leq i} x \phi(x)$

Theorem ([Lovász, 1967])

$\mathcal{A} \cong \mathcal{B} \iff |Hom(\mathcal{C}, \mathcal{A})| = |Hom(\mathcal{C}, \mathcal{B})| \; \forall$ *finite* $\mathcal{C}$

Theorem ([Dvořák, 2009])

$\mathcal{A} \equiv_{\mathcal{V}^k(\#)} \mathcal{B} \iff |Hom(\mathcal{C}, \mathcal{A})| = |Hom(\mathcal{C}, \mathcal{B})| \; \forall$ *finite* $\mathcal{C}$ *w/* $tw(\mathcal{C}) < k$

Theorem ([Grohe, 2020])

$\mathcal{A} \equiv_{QR_n(\#)} \mathcal{B} \iff |Hom(\mathcal{C}, \mathcal{A})| = |Hom(\mathcal{C}, \mathcal{B})| \; \forall$ *finite* $\mathcal{C}$ *w/* $td(\mathcal{C}) \leq n$

No-go without counting

Proposition ([Atserias et al., 2021])

There is no class of graphs $\mathcal{F}$ such that either of these hold:

$$A \equiv^{\mathcal{V}^k} B \iff |Hom(F, A)| = |Hom(F, B)| \ \forall F \in \mathcal{F}$$
$$A \equiv^{\mathcal{V}^k} B \iff |Hom(A, F)| = |Hom(B, F)| \ \forall F \in \mathcal{F}$$

Proposition ([Atserias et al., 2021])

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Solution using bisimulation

Instead of bijection between Hom sets, a "bisimulation" between Hom sets:

$$\mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \Leftrightarrow \quad \begin{tikzcd} & S_{\mathcal{C}} \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathrm{Hom}(\mathcal{C}, \mathcal{A}) & & \mathrm{Hom}(\mathcal{C}, \mathcal{B}) \end{tikzcd}$$

for all finite $\mathcal{C}$ w/ parameter $\leq k$ and $\exists S_{\mathcal{C}}$ and F, G (with some conditions)

Motivated by the framework of **Spoiler-Duplicator game comonads**

Spoiler-Duplicator games

Ehrenfeucht-Fraïssè game

- In every round i, of the n-round game $\mathbf{EF}_n(\mathcal{A}, \mathcal{B})$:
  - Spoiler chooses an element $a_i \in \mathcal{A}$ or $b_i \in \mathcal{B}$
  - Duplicator responds with $b_i \in \mathcal{B}$ or $a_i \in \mathcal{A}$
- Duplicator wins round i if $\gamma_i = \{(a_j, b_j) \mid j \leq i\}$ is a partial isomorphism
- If Duplicator has a winning response to every Spoiler move, Duplicator has a winning strategy.

Theorem ([Ehrenfeucht, 1961, Fraïssé, 1954])

Duplicator has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ *iff* $\mathcal{A} \equiv^{QR_n} \mathcal{B}$

Variants of EF game

One sided variant $\exists^+ \mathbf{EF}_n(\mathcal{A}, \mathcal{B})$:

- Spoiler only plays in $\mathcal{A}$, Duplicator responds in $\mathcal{B}$
- Winning condition: weakened form partial isomorphism to partial homomorphism
- Captures preservation in $\exists^+ QR_n$,

$$\mathcal{A} \Rrightarrow^{\exists^+ QR_n} \mathcal{B} \Leftrightarrow \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi \text{ for } \phi \in \exists^+ QR_n$$

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Bijection variant $\#\mathbf{EF}_n(\mathcal{A}, \mathcal{B})$

- Duplicator chooses a bijection $f : A \to B$
- Spoiler chooses $a_i \in \mathcal{A}$ or $b_i \in \mathcal{B}$
- Duplicator responds $f(a_i) \in \mathcal{B}$ or $f^{-1}(b_i) \in \mathcal{A}$
- Captures equivalence in $QR_n(\#)$: $\mathcal{A} \equiv^{QR_n(\#)} \mathcal{B}$

Spoiler-Duplicator games

Immerman's k-pebble game

- Spoiler and Duplicator each have k pebbles. On each round:
  - Spoiler places his pebble $p \in \mathbf{k}$ on $a_i \in \mathcal{A}$ or $b_i \in \mathcal{B}$
  - Duplicator places her corresponding pebble $p \in \mathbf{k}$ on $b_i \in \mathcal{B}$ or $a_i \in \mathcal{A}$

  Duplicator wins the round if the relation of all pebbled elements $\gamma = \{(a, b) \mid p \in \mathbf{k} \text{ w/ } p \text{ pebbling } a \in \mathcal{A}, b \in \mathcal{B}\}$ is a partial isomorphism

Theorem ([Immerman, 1982])

Duplicator has a winning strategy in $\mathrm{Peb}_k(\mathcal{A}, \mathcal{B})$ *iff* $\mathcal{A} \equiv^{\mathcal{V}^k} \mathcal{B}$

- One sided variant $\exists\mathbf{Peb}_k(\mathcal{A}, \mathcal{B})$: $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{V}^k} \mathcal{B}$ (preservation $\exists^+$)
- Bijection variant $\#\mathbf{Peb}_k(\mathcal{A}, \mathcal{B})$: $\mathcal{A} \equiv^{\mathcal{V}^k(\#)} \mathcal{B}$

Ehrenfeucht-Fraïssè comonad

Comonad on Σ is a triple $(\mathbb{E}_n, \varepsilon, ()^*)$

Given a σ-structure $\mathcal{A}$, we can create σ-structure on the set of Spoiler moves $\mathbb{E}_n A$ in $\exists^+\mathbf{EF}_n(\mathcal{A}, \cdot)$, i.e. non-empty sequences of elements in A of length $\leq n$

Let $\varepsilon_\mathcal{A} : \mathbb{E}_n\mathcal{A} \to \mathcal{A}$ return the last move of the play $[a_1, \ldots, a_m] \mapsto a_m$.

$$R^{\mathbb{E}_n\mathcal{A}}(s_1, \ldots, s_r) \Leftrightarrow s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \text{ for } i, j \in [\mathrm{r}]$$
$$\text{and } R^\mathcal{A}(\varepsilon_\mathcal{A}(s_1), \ldots, \varepsilon_\mathcal{A}(s_r))$$

For $f : \mathbb{E}_n\mathcal{A} \to \mathcal{B}$ define $f^* : \mathbb{E}_n\mathcal{A} \to \mathbb{E}_n\mathcal{B}$ recursively:

$$f^*(s[a]) = f^*(s)[f(a)]$$

EF comonad to game

- Functions $f : \mathbb{E}_n A \to B$ are Duplicator's strategies in $\exists^+ \mathbf{EF}_n(\mathcal{A}, \mathcal{B})$
- Chose relations so that σ-morphisms $f : \mathbb{E}_n \mathcal{A} \to \mathcal{B}$ are Duplicator's **winning** strategies.
- Coextension $f^* : \mathbb{E}_n \mathcal{A} \to \mathbb{E}_n \mathcal{B}$ models history preservation of the game

Theorem ([Abramsky and S, 2018])

The following are equivalent:

1. *Duplicator has a winning strategy in* $\exists^+ \mathrm{EF}_n(\mathcal{A}, \mathcal{B})$
2. *There exists a Kleisli morphism* $f : \mathbb{E}_n \mathcal{A} \to \mathcal{B}$

Pebbling comonad

- Similar construction for k-pebble game, where $\mathbb{P}_k\mathcal{A}$ is sequences $[(p_1, a_1), \ldots, (p_n, a_n)]$ with $p_i \in \mathbf{k}$
- Additional active pebble condition in defining $R^{\mathbb{P}_k\mathcal{A}}$
- $\mathbb{P}_k\mathcal{A}$ necessarily infinite, i.e. not comonad over Σ_f
- Bound length of sequences by $\leq n$, $\mathbb{P}_{k,n}$ obtains a comonad over Σ_f
- First game comonad discovered from Abramksy, Dawar, and Wang LICS 2017

Game comonad power theorems

Theorem ([Abramsky et al., 2017, Abramsky and S, 2018])

For all game comonads $\mathbb{C}_k$ *and* $\mathcal{A}, \mathcal{B} \in \Sigma_f$

$$\mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \rightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \textit{Duplicator wins } \exists^+\mathsf{G}_k(\mathcal{A}, \mathcal{B})$$
$$\mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \leftrightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \textit{Duplicator wins } \mathsf{G}_k(\mathcal{A}, \mathcal{B})$$
$$\mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} \Leftrightarrow \mathcal{A} \cong_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \textit{Duplicator wins } \#\mathsf{G}_k(\mathcal{A}, \mathcal{B})$$

The $\rightarrow_k^{\mathbb{C}}$ and $\cong_k^{\mathbb{C}}$ arise from $\mathbf{Kl}(\mathbb{C}_k)$:

- $\rightarrow_k^{\mathbb{C}}$ if the there exists a Kleisli morphism $f : \mathbb{C}_k\mathcal{A} \rightarrow \mathcal{B}$
- $\cong_k^{\mathbb{C}}$ if there are pairs of Kleisli morphisms $f : \mathbb{C}_k\mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathbb{C}_k\mathcal{B} \rightarrow \mathcal{A}$ that are inverses

The $\leftrightarrow_k^{\mathbb{C}}$ arises from a notion of open map bisimulation [Joyal et al., 1996] in the category of coalgebras $\mathbf{EM}(\mathbb{C}_k)$

Eilenberg-Moore category of coalgebras

Coalgebras are morphisms $\alpha : \mathcal{A} \to \mathbb{E}_n\mathcal{A}$ satisfying the equations:

$$\epsilon_{\mathcal{A}} \circ \alpha = \mathsf{id}_{\mathcal{A}} \qquad \mathbb{C}_k\alpha \circ \alpha = \delta_{\mathcal{A}} \circ \alpha$$

We can define the Eilenberg-Moore category $\mathbf{EM}(\mathbb{E}_n)$:

- Objects are coalgebras $(\mathcal{A}, \alpha : \mathcal{A} \to \mathbb{E}_n\mathcal{A})$
- Morphisms are commuting squares:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathbb{E}_n\mathcal{A} \arrow[d, "\mathbb{E}_n f"] \\ \mathcal{B} \arrow[r, "\beta"] & \mathbb{E}_n\mathcal{B} \end{tikzcd}$$

Tree-depth, Forest covers and $\mathbb{E}_n$

Proposition ([Abramsky and S, 2018])

Category of $\mathbb{E}_n$-coalgebras $\mathsf{EM}(\mathbb{E}_n)$ *isomorphic to a category of n-height forest covers* $\Sigma_n^{\mathbb{E}}$

- Objects are $(\mathcal{A}, \leq)$ where $\mathcal{A} \in \Sigma$ and $\leq$ is a forest order on A:
  - All the elements below an element $a \in \mathcal{A}$ form a chain.
  - If a, a' are related in $\mathcal{A}$, then $a \leq a'$ or $a' \leq a$.
- Morphisms are homomorphisms that preserve the covering relation

Corollary ([Abramsky and S, 2018])

Coalgebra $\mathcal{A} \to \mathbb{E}_n\mathcal{A}$ *iff* $td(\mathcal{A}) \leq n$

Bisimulation $\leftrightarrow_n^{\mathbb{E}}$

We can pick out subcategory of paths in $\Sigma^{\mathbb{E}}$ as objects $(P, \leq)$ where $\leq$ is a linear order

Paths in $(\mathcal{A}, \leq)$ can be seen as $\Sigma_n^{\mathbb{E}}$-morphisms $i : (P, \leq) \rightarrowtail (\mathcal{A}, \leq)$ which are embeddings (as a Σ morphism)

Bisimulation $\leftrightarrow_k^{\mathbb{E}}$

Given σ-structures $\mathcal{A}, \mathcal{B}$, we say $\mathcal{A} \leftrightarrow_n^{\mathbb{E}} \mathcal{B}$ iff there exists a span:

$$\begin{tikzcd} & (\mathcal{R}, \leq_{\mathcal{R}}) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (\mathbb{E}_n\mathcal{A}, \sqsubseteq) & & (\mathbb{E}_n\mathcal{B}, \sqsubseteq) \end{tikzcd}$$

Where f, g are:

- Pathwise embedding: $e : (P, \leq) \rightarrowtail (\mathcal{R}, \leq_{\mathcal{R}}) \Rightarrow f \circ e : (P, \leq) \rightarrowtail (\mathbb{E}_n\mathcal{A}, \sqsubseteq)$
- Open: If a path can be extended to a larger path in the image, then the preimage of the path can also be extended.

Path-lifting property

$$\begin{tikzcd}
(\mathsf{P}, \leq) \arrow[r, tail] \arrow[d, tail] & (\mathsf{Q}, \leq) \arrow[d, tail] \\
(\mathcal{R}, \leq_{\mathcal{R}}) \arrow[r, "f"'] & (\mathbb{E}_n \mathcal{A}, \sqsubseteq)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{P}, \leq) \arrow[r, tail] \arrow[d, tail] & (\mathsf{Q}, \leq) \arrow[d, tail] \arrow[dl, tail, dashed] \\
(\mathcal{R}, \leq_{\mathcal{R}}) \arrow[r, "f"'] & (\mathbb{E}_n \mathcal{A}, \sqsubseteq)
\end{tikzcd}$$

This is modified notion of open map bisimulation, used in categories of transition systems, event structures, etc. [Joyal et al., 1996]

Fully explicated and generalized to "arboreal categories" [Abramsky and Reggio, 2021]

Treewidth and $\mathbb{P}_k$

Proposition ([Abramsky et al., 2017])

Category of $\mathbb{P}_k$-coalgebras $\mathrm{EM}(\mathbb{P}_k)$ *isomorphic to a category of k-pebble forest covers* $\Sigma_n^{\mathbb{E}}$

- Objects are $(\mathcal{A}, \leq, p : \mathcal{A} \to \mathbf{k})$ with $(\mathcal{A}, \leq)$ a forest cover and p satisfying some conditions.
- Morphisms are morphisms of forest covers that preserver the pebbling function.

Corollary ([Abramsky et al., 2017])

Coalgebra $\mathcal{A} \to \mathbb{P}_k\mathcal{A}$ *iff* $tw(\mathcal{A}) < k$

Categorified Lovász using comonads

Theorem (Dawar,Jakl,Reggio 2021)

If $\mathbb{C}$ is a comonad on Σ, then:

$$\mathcal{A} \cong^{\mathbb{C}} \mathcal{B} \Leftrightarrow Hom_{\Sigma_F}(\mathcal{C}, \mathcal{A}) \cong Hom_{\Sigma_F}(\mathcal{C}, \mathcal{A})$$

for all finite coalgebras $\mathcal{C} \to \mathbb{C}\mathcal{C}$

Applied to $\mathbb{E}_n$ and $\mathbb{P}_k$ yields new proofs for Dvořák 2009 and Grohe 2020

Not direct applications, must be explicit about equality in signature

- For $\mathbb{P}_k$, use of the approximate comonads $\mathbb{P}_{k,n}$

Also prove a new Lovász result for graded modal logic using $\mathbb{M}_k$

"One-sided" Lovász-type result

Proposition

If $\mathbb{C}$ comonad on Σ restricting to Σ_f, then:

$$\mathcal{A} \to^{\mathbb{C}} \mathcal{B} \Leftrightarrow Hom_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \to Hom_{\Sigma_f}(\mathcal{C}, \mathcal{B})$$

for all finite coalgebras $\mathcal{C} \to \mathbb{C}\mathcal{C}$

For $\mathbb{E}_n$, get the equivalence

$$\mathcal{A} \Rrightarrow^{\exists^+ QR_n} \mathcal{B} \Leftrightarrow \mathcal{C} \to \mathcal{A} \Rightarrow \mathcal{C} \to \mathcal{B}$$

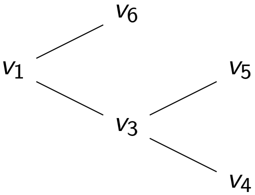
for all finite $\mathcal{C}$ w/ $\mathrm{td}(\mathcal{C}) \leq n$

Used in Rossman's HPT paper, CSPs, Datalog, etc.

Spoiler's winning strategy in the $\exists^{+}\mathbf{EF}_n(\mathcal{A}, \mathcal{B})$ is given by a primitive $\exists^{+}$-sentence ϕ such that $\mathcal{A} \vDash \phi$, $\mathcal{B} \nvDash \phi$ and $\mathrm{qr}(\phi) \leq n$

We can view such sentences as $\mathbb{E}_n$-coalgebras, consider $n = 3$

$$\exists v_1(\exists v_3(\phi_a(v_1, v_3) \wedge \exists v_5 \phi_b(v_1, v_3, v_5) \wedge \exists v_4 \phi_c(v_1, v_3, v_4)) \wedge \exists v_6(\phi_d(v_1, v_6)))$$



$\mathcal{A} \vDash \phi \Leftrightarrow \mathcal{C}[\phi] \to \mathcal{A}$ in $\Sigma_3^{\mathbb{E}}$

There exists a finite coalgebra $\mathcal{C}[\phi]$ such that there is no function $\mathrm{Hom}(\mathcal{C}[\phi], \mathcal{A}) \to \mathrm{Hom}(\mathcal{C}[\phi], \mathcal{B})$

Applying "one-sided" Lovász result, $\mathcal{A} \not\to_3^{\mathbb{E}} \mathcal{B}$

Bisimulation completes the picture

Theorem (+Conjecture)

For all game comonads $\mathbb{C}_k$ and finite coalgebras $\mathcal{C} \to \mathbb{C}_k\mathcal{C}$:

$$\mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \rightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow Hom_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \rightarrow Hom_{\Sigma_f}(\mathcal{C}, \mathcal{B})$$

$$\mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \leftrightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \exists \text{ span } Hom_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \leftarrow S \rightarrow Hom_{\Sigma_f}(\mathcal{C}, \mathcal{B})?$$

$$\mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} \Leftrightarrow \mathcal{A} \cong_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow Hom_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \cong Hom_{\Sigma_f}(\mathcal{C}, \mathcal{B})$$

Removing naturality

Theorem (Yoneda)

$A \cong B \iff Hom(-, A) \cong_{[\Sigma_f^{op}, \mathsf{Set}]} Hom(-, B)$

Yoneda embedding is fully faithful and so reflects isomorphisms

Theorem (Lovász 1967)

$A \cong B \iff Hom(F, A) \cong_{\mathsf{Set}} Hom(F, B) \; \forall F \in \Sigma_f$

Lovász: For finite relational structures Σ_f, isomorphism between the sets suffices

Before bisimulation analog of Lovász, is there a bisimulation analog of Yoneda?

Bisimulation of presheaves

Let $\mathcal{T}_n^{\mathbb{E}}$ be the subcategory of n-height forest covers $\Sigma_n^{\mathbb{E}}$ with pathwise embeddings. $F_n\mathcal{A} = (\mathbb{E}_n\mathcal{A}, \sqsubseteq)$. Adapting lemma 16 of [Joyal et al., 1996]

Bisimulation of presheaves

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$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ F_n\mathcal{A} & & F_n\mathcal{B} \end{tikzcd}$$

$$\Leftrightarrow$$

$$\begin{tikzcd} & \hat{S}(-) \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathrm{Hom}_{\mathcal{T}_n^{\mathbb{E}}}(-, F_n\mathcal{A}) & & \mathrm{Hom}_{\mathcal{T}_n^{\mathbb{E}}}(-, F_n\mathcal{B}) \end{tikzcd}$$

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$$\Leftrightarrow$$

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Where F, G are:

- Componentwise epimorphic, for all C, $F_C : \hat{S}(C) \to \mathrm{Hom}(C, F_nA)$ is a surjective function.
- Open maps in a topos [Joyal and Moerdijk, 1994]

Open maps in a topos

Dropping subscript $\mathrm{Hom}_{\mathcal{T}_n^{\mathbb{E}}}(-,-) = \mathrm{Hom}(-,-)$

For all path embeddings $i : P \rightarrowtail Q$, the following is an quasi-pullback square:

$$
\begin{tikzcd}
\hat{S}(Q) \arrow[r, "\hat{S}(i)"] \arrow[d, "F_Q"'] & \hat{S}(P) \arrow[d, "F_P"] \\
\mathrm{Hom}(Q, F_n A) \arrow[r, "\mathrm{Hom}(i, F_n A)"'] & \mathrm{Hom}(P, F_n A)
\end{tikzcd}
$$

The mediating morphism

$$\hat{S}(Q) \to \hat{S}(P) \times_{\mathrm{Hom}(P, F_n A)} \mathrm{Hom}(Q, F_n A)$$

is a surjective function.

Conclusion

Good evidence for bisimulation between hom sets as a way of achieving a Lovász result for logics without counting

Completing this picture

$$\mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \to_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \to \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{B})$$

$$\mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \leftrightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \exists \text{ span } \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \leftarrow S \to \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{B})?$$

$$\mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} \Leftrightarrow \mathcal{A} \cong_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{A}) \cong \mathrm{Hom}_{\Sigma_f}(\mathcal{C}, \mathcal{B})$$

As "removing natuality" from open maps of presheaves inducing open maps of structures

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