A categorical approach to constraint satisfaction

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joint work with M. Wrochna, and others

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the constraint satisfaction problem

(Contraint Satisfaction Problem) Given a list of constraints over some domain D involving variables from V where each constraint is of the form $(v_1, \dots, v_k) \in R$ for some $R \subseteq D^k$, decide whether there is a satisfying assignment $V \to D$.

What is the complexity of CSP depending on the shape of relations R?

Examples

- Sat, 3-Sat, Horn-Sat, etc.;
- graph colouring (3-colouring);
- linear equations (over a finite field, $\mathbb{Z}$, $\mathbb{Q}$, etc.);
- linear programming (convex constraints over $\mathbb{Q}$);

constraint satisfaction problems

Fix a relational structure $\mathbf{A}$. The CSP($\mathbf{A}$) is the following (decision) problem. Given another structure $\mathbf{I}$ in the same signature, decide whether there is a homomorphism $\mathbf{I} \to \mathbf{A}$.

A relational structure is a tuple $\mathbf{A} = (A; R^{\mathbf{A}}, S^{\mathbf{A}}, \dots)$ such that $R \subseteq A^{\operatorname{ar} R}$, etc. The set of symbols $R, S, \dots$ together with their arities is its signature.

an old story

The dichotomy [Feder, Vardi, “98; Bulatov-Zhuk, ‘17].
For every finite structure $\mathbf{A}$, either $\mathrm{CSP}(\mathbf{A})$ is polynomial time solvable, or it is NP-complete.

- dichotomy of Boolean CSPs [Scheafer, “78]
- dichotomy of (undirected) graph CSPs [Hell, Nešetřil, “90]
- the dichotomy conjecture [Feder, Vardi, “98]
- pol-inv Galois correspondence [Cohen, Gyssens, Jeavons, “97]
- HSP closure (the hardness side) [Bulatov, Jeavons, Krokhin, ‘05]
- polynomial time algorithms [Bulatov, ‘17; Zhuk, ‘17]

a new story

reductions

Assume that **A** and **B** are two (finite) relational structures.

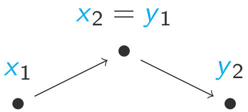
A reduction from CSP($\rho\mathbf{A}$) to CSP($\mathbf{A}$) is a mapping

$$\lambda\colon \text{ structures similar to } \rho\mathbf{A} \to \text{ structures similar to } \mathbf{A}$$

such that

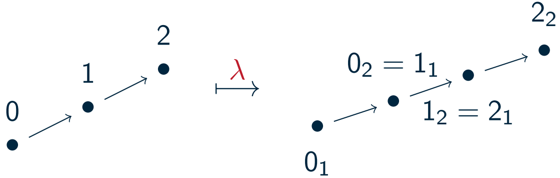
$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}.$$

a gadget reduction λ



$$\phi(x_1, x_2, y_1, y_2) = (x_1, x_2) \in E \wedge (y_1, y_2) \in E \wedge x_2 = y_1.$$

Example



a pp-power ρ

$\rho\mathbf{A}$ is a pp-power of $\mathbf{A}$.

Concretely, $\rho\mathbf{A} = (A^2; E^{\rho\mathbf{A}})$ where

$$\begin{aligned} ((a_1, a_2), (b_1, b_2)) \in E^{\rho\mathbf{A}} & \\ \text{iff} \quad & \mathbf{A} \models \phi(a_1, a_2, b_1, b_2) \\ \text{iff} \quad & (a_1, a_2) \in E^{\mathbf{A}} \wedge (b_1, b_2) \in E^{\mathbf{A}} \wedge a_2 = b_1. \end{aligned}$$

Observation

$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}$$

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algebraic approach in a nutshell

Theorem [Bulatov, Jeavons, Krokhin, '05; Barto, **O**, Pinsker, '17]
The following are equivalent for any finite relational structures $\mathbf{A}$, $\mathbf{B}$:

1. there is a gadget reduction from $\mathrm{CSP}(\mathbf{B})$ to $\mathrm{CSP}(\mathbf{A})$;
2. $\mathbf{B}$ is homomorphically equivalent to a pp-power of $\mathbf{A}$;
3. there is a minion (h1 clone) homomorphism from $\mathrm{pol}(\mathbf{A})$ to $\mathrm{pol}(\mathbf{B})$.

definitions

minions

An abstract minion is a functor $\mathscr{M}\colon \mathbf{Set}_{\text{fin}} \to \mathbf{Set}$.

A minion homomorphism is a natural map between functors, i.e., a collection of maps $\xi_A\colon \mathscr{M}(A) \to \mathscr{N}(A)$ such that

$$\begin{array}{ccc} \mathscr{M}(A) & \xrightarrow{\mathscr{M}(\pi)} & \mathscr{M}(B) \\ \downarrow{\scriptstyle \xi_A} & & \downarrow{\scriptstyle \xi_B} \\ \mathscr{N}(A) & \xrightarrow{\mathscr{N}(\pi)} & \mathscr{N}(B) \end{array}$$

commutes for all maps $\pi\colon A \to B$.

polymorphisms

A polymorphism of $\mathbf{A}$ of arity n is a map $f\colon A^n \to A$ such that for each relation $R^{\mathbf{A}}$ and all $r_1, \dots, r_n \in R^{\mathbf{A}}$, $i \in N$ we have

$$f(\mathbf{r}_1, \dots, \mathbf{r}_n) \in R^{\mathbf{A}}$$

where f is applied coordinate-wise.

The polymorphism minion of $\mathbf{A}$, denoted by $\mathrm{pol}(\mathbf{A})$, maps the set $[n]$ to the set of all polymorphisms of $\mathbf{A}$ of arity n.

The image of a map $\pi\colon [m] \to [n]$ under $\mathrm{pol}(\mathbf{A})$ is the operation of ‘taking minors’, i.e., $f \mapsto f^\pi$ where

$$f^\pi(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(m)}).$$

proof?

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3. there is a minion homomorphism from $\mathrm{pol}(\mathbf{A})$ to $\mathrm{pol}(\mathbf{B})$.

(3)$\to$(1) given a minion homomorphism $\mathrm{pol}(\mathbf{A}) \to \mathrm{pol}(\mathbf{B})$ we have a chain of gadget reductions

$$\mathrm{CSP}(\mathbf{B}) \to \mathrm{PLC}(\mathrm{pol}(\mathbf{B})) \to \mathrm{PLC}(\mathrm{pol}(\mathbf{A})) \to \mathrm{CSP}(\mathbf{A}).$$

a middle problem

Label cover

Is a binary CSP, where variables $v_1, \dots, v_n$ can have different domains $D_1, \dots, D_n$ and constraints are of the form $v_j = \pi(v_i)$ for some $\pi \colon D_i \to D_j$.

To any minion we associate a promise problem $\mathrm{PLC}(\mathscr{M})$. Given an LC instance, output

- yes if it a solvable LC instance,
- no if there is no assignment s of values to variables with $s(v_i) \in \mathscr{M}(D_i)$ such that
$$s(v_j) = \mathscr{M}(\pi)(s(v_i))$$
for each constraint $v_j = \pi(v_j)$.

This problem falls into a wider scope of promise constraint satisfaction problems.

a theorem

Theorem [Barto, Bulín, Krokhin, **O**, ‘19]
CSP($\mathbf{A}$) is log-space equivalent to PLC(pol(A)).

The two reductions required are gadget reductions!

proof?

A gadget reduction reduction λ can be classified by its corresponding pp-power ρ so that

$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}$$

Note. $\mathbf{B} \mapsto \mathrm{pol}(\mathbf{A}, \mathbf{B})$ is the ‘pp-power’ for the reduction from $\mathrm{PLC}(\mathrm{pol}(\mathbf{A}))$ to $\mathrm{CSP}(\mathbf{A})$

$\mathrm{pol}(\mathbf{A}, \mathbf{B})$ denotes the polymorphism minion from $\mathbf{A}$ to $\mathbf{B}$. A polymorphism from $\mathbf{A}$ to $\mathbf{B}$ is a homomorphism $\mathbf{A}^n \to \mathbf{B}$.

proof...

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The following are equivalent for any finite relational structures $\mathbf{A}$, $\mathbf{B}$:

1. there is a gadget reduction from $\mathrm{CSP}(\mathbf{B})$ to $\mathrm{CSP}(\mathbf{A})$;
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3. there is a minion homomorphism from $\mathrm{pol}(\mathbf{A})$ to $\mathrm{pol}(\mathbf{B})$.

(2)$\to$(3) is essentially given by the following

Lemma [Wrochna, Živný, '20]
If ρ preserves products, then there is a minion homomorphism

$$\mathrm{pol}(\rho\mathbf{A}) \to \mathrm{pol}(\mathbf{A}).$$

a question

Algebraic approach classifies gadget reductions.

Question

Is there a more general class of CSP reductions that can be classified by a similar (but stronger) theory?

E.g., also include reductions that replace tuple of variables by a tuple of variables.

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