Quantum isomorphism of graphs: An overview

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Structure meets Power

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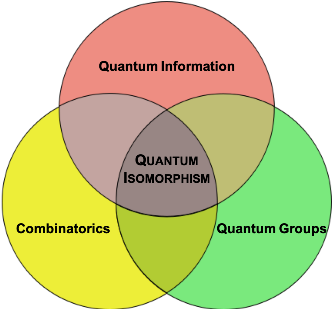


Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs.
Mančinska, Roberson.
Proceedings of FOCS'20.

Nonlocal games and quantum permutation groups.
Lupini, Mančinska, and Roberson.
Journal of Functional Analysis, **279(5)**:108592, 2020.

Quantum and non-signalling graph isomorphisms.
Atserias, Mančinska, Roberson, Šámal, Severini, and Varvitisiotis.
Journal of Combinatorial Theory, Series B, **136**:289–328, 2019.
Proceedings of ICALP'17 , LIPIcs **80**, 76:1–76:14, 2017.

This talk

PART I

Quantum isomorphism and different ways to think about it:

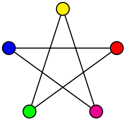
- Nonlocal games
- Matrix formulations
- Homomorphism counts

PART II

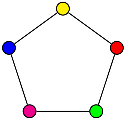
Elements of the proof:

- Intertwiners of quantum groups
- Bi-labeled graphs
- Homomorphism matrices

Graph isomorphism



$\cong$



A map $f : V(G) \to V(H)$ is an **isomorphism** from G to H if

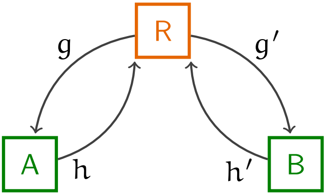
- f is a bijection and
- $g \sim g'$ if and only if $f(g) \sim f(g')$.

If such a map exists, we say that G and H are **isomorphic** and write $G \cong H$.

Matrix formulation: $PA_GP^\dagger = A_H$ for some **permutation** matrix P

(G, H)-Isomorphism Game

Intuition: Alice and Bob want to convince a referee that $G \cong H$.



- To win players must reply h, h' such that $\mathrm{rel}(h, h') = \mathrm{rel}(g, g')$
- No communication during game

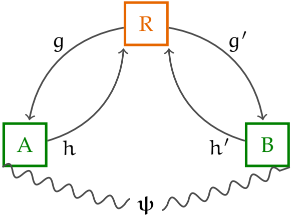
Fact. $G \cong H \Leftrightarrow$ **Classical** players can win the game with certainty

Def. (Quantum isomorphism)
We say that $G \cong_{qc} H$ if **quantum**[1] players can win the game with certainty.

[1] We work in the **commuting rather than the tensor-product model.**

Quantum commuting strategies

$G \cong_{qc} H :=$ **Quantum** players can win the (G, H)-isomorphism game

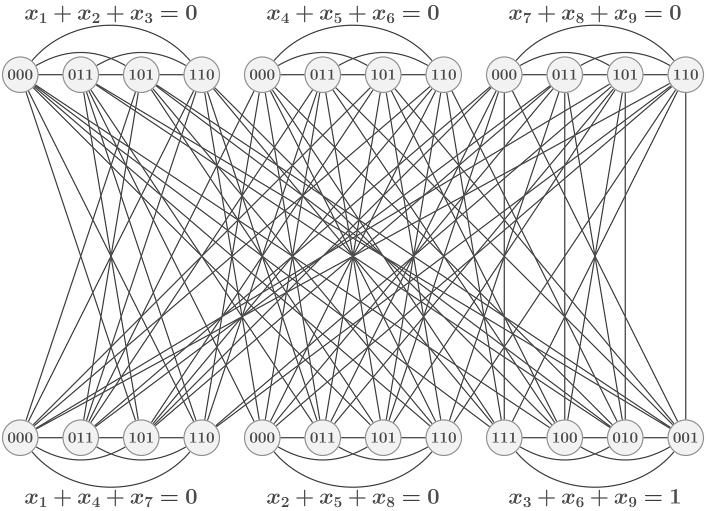


- Alice and Bob share a quantum state ψ
  ψ is a unit vector in a Hilbert space $\mathcal{H}$
- Upon receiving g, Alice performs a local measurement $\mathcal{E}_g$ to get $h \in V(H)$
  $\mathcal{E}_g = \{E_{gh} \in \mathcal{B}(\mathcal{H}) : h \in V(H)\}$ where
  $$E_{gh} \succeq 0, \quad \sum_h E_{gh} = I.$$
- Bob measures with $\mathcal{F}_{g'}$
- E_{gh} and $F_{g'h'}$ commute

The probability that players respond with h, h' on questions g, g' is

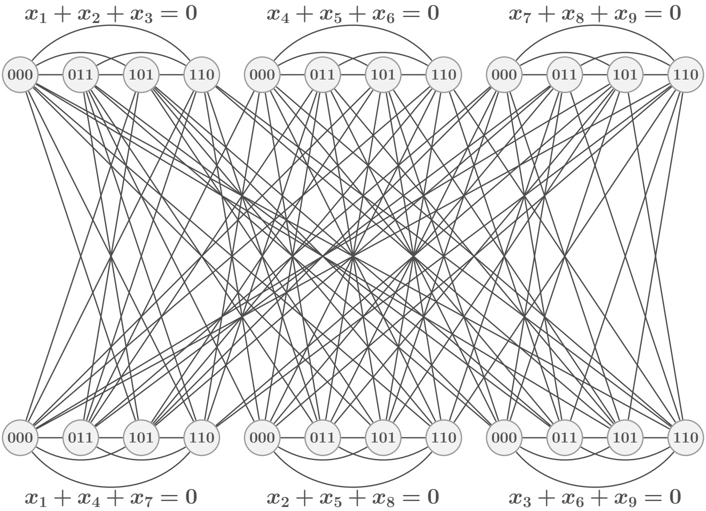
$$p(h, h'|g, g') = \langle \psi, \left(E_{gh} F_{g'h'}\right) \psi \rangle$$

Example: $G \not\cong H$ but $G \cong_{qc} H$



Construction based on reduction from linear system games.

Example: $\mathsf{G} \not\cong \mathsf{H}$ but $\mathsf{G} \cong_{\mathrm{qc}} \mathsf{H}$



Construction based on reduction from linear system games.

Undecidability

Cor. Given two graphs G and H it is undecidable to test whether they are quantum isomorphic.

Quantum isomorphism and quantum groups

Def. A matrix $\mathcal{P} = (p_{ij})$ whose entries are elements of a C^*-algebra is a **quantum permutation matrix** (QPM), if

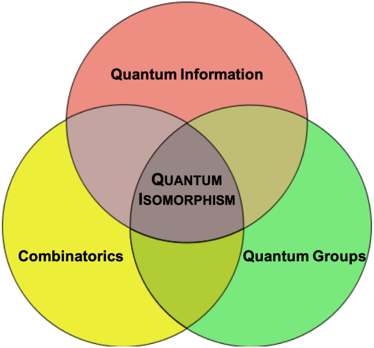
- p_{ij} is a projection, i.e., $p_{ij}^2 = p_{ij} = p_{ij}^*$ for all i, j
- $\sum_k p_{ik} = \mathbf{1} = \sum_\ell p_{\ell j}$ for all i, j

Remark. A QPM with entries from $\mathbb{C}$ is a permutation matrix.

Thm. (Lupini, M., Roberson)

$G \cong_{qc} H \quad \Leftrightarrow \quad \mathcal{P} A_G \mathcal{P}^\dagger = A_H$ for some **quantum permutation matrix** $\mathcal{P}$

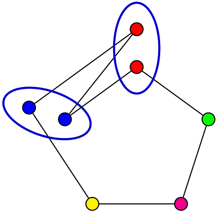
Can we describe quantum isomorphism in combinatorial terms?



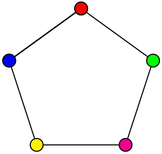
Graph homomorphisms

Def. A map $\varphi : V(F) \to V(G)$ is a **homomorphism** from F to G if $\varphi(u) \sim \varphi(v)$ whenever $u \sim v$.

Example



$C_7 \to C_5$



hom(F, G) := # of homomorphisms from F to G.

Counting homomorphisms

Theorem. (Lovász, 1967)
Homomorphism counts determine a graph up to isomorphism, i.e.

$$G \cong H \;\Leftrightarrow\; \mathrm{hom}(F, G) = \mathrm{hom}(F, H) \text{ for all graphs } F.$$

Theorem. (M., Roberson)
$G \cong_{qc} H \;\Leftrightarrow\; \mathrm{hom}(F, G) = \mathrm{hom}(F, H)$ for all **planar** graphs F.

Context: Homomorphism counting

Thm. (Lovász, 1967)
$G \cong H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for **all graphs** F

Thm. (M., Roberson, 2019)
$G \cong_{qc} H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for all **planar** graphs F

Folklore.
G and H cospectral $\Leftrightarrow \hom(F, G) = \hom(F, H)$ for all **cycles** F

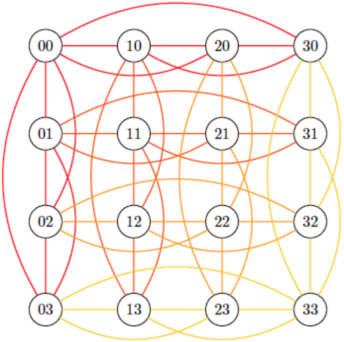
Thm. (Dvořák, 2010; Dell, Grohe, Rattan, 2018)
$G \cong_f H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for all **trees** F
$G \cong_k H \Leftrightarrow \hom(F, G) = \hom(F, H)$ for all F of **treewidth** $\leqslant k$

Complexity: Except for the class of planar graphs, equality of homomorphism counts from all of the above graph classes can be tested in at worst quasi-polynomial time.

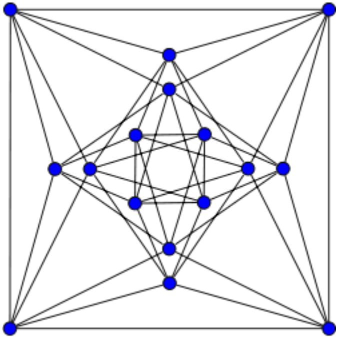
Application: Certificate for $G \not\cong_{qc} H$

Are these two graphs quantum isomorphic?

Rook graph



Shrikhande graph



Before: Difficult to prove that they are not quantum isomorphic.

Now: Only one (the Rook graph) contains K_4.

Part II
Elements of the proof

Thm. $G \cong_{qc} H \iff \mathrm{hom}(F, G) = \mathrm{hom}(F, H)$ for all **planar** graphs F

Main component of our proof: Provide a *combinatorial description* of the **intertwiners** of $\mathrm{Qut}(G)$.

The quantum automorphism group $\mathrm{Qut}(G)$

Definition. (Banica 2005)
$C(\mathrm{Qut}(G))$ is the universal C^*-algebra generated by elements u_{ij}, $i, j \in V(G)$, satisfying the following:

1. $\mathcal{U} = (u_{ij})$ is a quantum permutation matrix.
2. $A_G \mathcal{U} = \mathcal{U} A_G$.

The **quantum automorphism group** $\mathrm{Qut}(G)$ is given by $C(\mathrm{Qut}(G))$ together with the comultiplication

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

The matrix $\mathcal{U}$ is called the **fundamental representation** of $\mathrm{Qut}(G)$.

Intertwiners of $\mathrm{Qut}(G)$

- $\mathcal{U} = (u_{ij})$ - fundamental representation of $\mathrm{Qut}(G)$.
- $\left(\mathcal{U}^{\otimes k}\right)_{i_1 \ldots i_k, j_1 \ldots j_k} = u_{i_1 j_1} \ldots u_{i_k j_k}$

Definition. An (ℓ, k)-**intertwiner** of $\mathrm{Qut}(G)$ is a matrix T s.t.

$$\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k}.$$

$C_q^G = \{T : T \text{ is an } (\ell, k)\text{-intertwiner for some } \ell, k \in \mathbb{N}\}$.

C_q^G is closed under matrix product, tensor product, conjugate transposition, and linear combinations.

Examples of intertwiners of Qut(G)

$(1,1)$-**intertwiner:** $\mathcal{U}A_G = A_G\mathcal{U}$

$(1,2)$-**intertwiner:** $M(e_i \otimes e_j) = \delta_{ij}e_i$.

$(1,0)$-**intertwiner:** $U = \sum_{i=1}^n e_i$.

Theorem. (Chassaniol 2019)

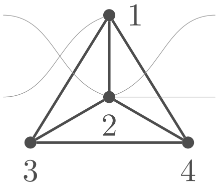
$$C_q^G = \langle U, M, A_G \rangle_{+,\circ,\otimes,*}$$

Bi-labeled graphs

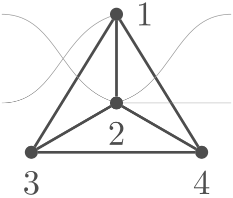
Def. (Lovász, Large Networks and Graph Limits)
An (ℓ, k)-**bi-labeled graph** is a triple $\vec{F} = (F, \vec{a}, \vec{b})$ where

- F is a graph
- $\vec{a} = (a_1, \ldots, a_\ell)$ and $\vec{b} = (b_1, \ldots, b_k)$ are tuples of vertices of F.

Example. $\vec{F} = \big(K_4, (2,1), (2,2)\big)$

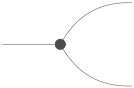


How to draw bi-labeled graphs



$$\vec{\mathsf{F}} = \left(\mathsf{K}_4, (2,1), (2,2)\right)$$

$\vec{U} = \left(K_1, (1), \varnothing\right)$



$\vec{M} = \left(K_1, (1), (1,1)\right)$

$\vec{A} = \left(K_2, (1), (2)\right)$

Homomorphism matrices

Let G be a graph and $\vec{F} = (F, (a), (b))$ an $(1,1)$-bi-labeled graph.

Def. (G-homomorphism matrix of $\vec{F}$)
For $u, v \in V(G)$, the uv-entry of the **homomorphism matrix** $T^{\vec{F}}$ is

$$\left|\{\text{homs } \varphi : F \to G \mid \varphi(a) = u,\ \varphi(b) = v\}\right|.$$

Example. $\vec{A} = (K_2, (1), (2))$ 

$$\left(T^{\vec{A}}\right)_{u,v} = \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

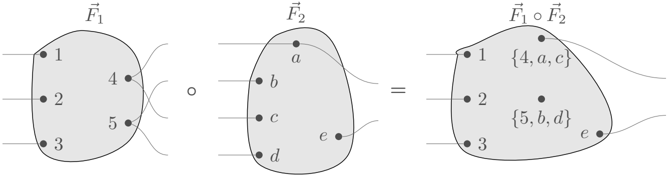
So $T^{\vec{A}} = A_G$. Similarly, $T^{\vec{U}} = U, \quad T^{\vec{M}} = M.$

Operations on bi-labeled graphs: Products

Thm. For a graph G and bi-labeled graphs $\vec{F}_1, \vec{F}_2$,

$$T^{\vec{F}_1} T^{\vec{F}_2} = T^{\vec{F}_1 \circ \vec{F}_2},$$

where $\vec{F}_1 \circ \vec{F}_2$ is defined as

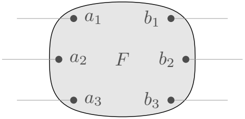


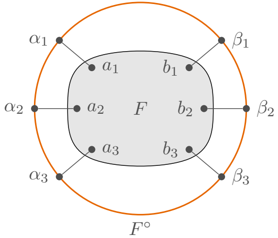
Planar bi-labeled graphs

Recall: Intertwiners of $\mathrm{Qut}(G) = \langle \mathcal{U}, \mathcal{M}, \mathcal{A}_G \rangle_{\circ,\otimes,*,\text{lin}}$

So we want to know what bi-labeled graphs are in $\langle \vec{\mathcal{U}}, \vec{\mathcal{M}}, \vec{\mathcal{A}} \rangle_{\circ,\otimes,*}$.

Def.



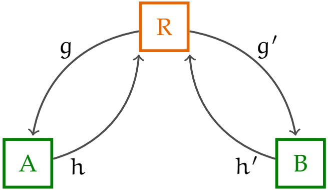


$\mathcal{P} = \{\vec{F} : F^\circ$ has planar embedding w/ **enveloping cycle** bounding outer face$\}$

Thm. (informal) Intertwiners of $\mathrm{Qut}(G)$ are given by the span of homomorphism matrices of planar bi-labeled graphs.

Summary

Graph isomorphism can be formulated in terms of a **nonlocal game**.



- $G \cong_{qc} H :=$ **Quantum** players can win the isomorphism game

Quantum isomorphisms have a rich mathematical structure:

- **Thm.** $G \cong_{qc} H \quad \Leftrightarrow \quad \mathcal{P} A_G \mathcal{P}^\dagger = A_H$ for some **quantum permutation matrix** $\mathcal{P}$
- **Thm.** $G \cong_{qc} H \quad \Leftrightarrow \quad \mathrm{hom}(F, G) = \mathrm{hom}(F, H)$ for all **planar** F

Thank you!