Traversal-invariant definability & logarithmic-space computation

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Properties definable in first-order logic (FO)

Definition: A property of graphs is *elementary* if it is FO definable.

- An edge relation E on a vertex set V forms a *simple graph* iff

$$\forall x, y, z \in V, [E(x, y) \rightarrow E(y, x)] \wedge \neg E(z, z).$$

- A binary relation < on a set S is a *linear order* iff

$$\forall x, y, z \in S, z \nless z \wedge [x \neq y \rightarrow (x < y \vee y < x)] \wedge [x < y < z \rightarrow x < z].$$

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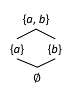
Finite Boolean algebras (under inclusion)



Define: The *parity* of a finite Boolean algebra of size 2^n is n mod 2.

Claim: The parity of a finite Boolean algebra is not elementary.

Proof: Let T be the theory of atomic Boolean algebras formulated in the language of a binary subset relation, $\subseteq$. Every finite Boolean algebra $\langle 2^n; \subseteq\rangle$ is atomic.

- Now suppose toward a contradiction that $\langle 2^n; \subseteq\rangle \vDash \theta \Leftrightarrow n$ even.

Both θ and $\neg\theta$ have arbitrarily large finite models of T, which means they both have infinite models of T by compactness. But all infinite models of T are elementarily equivalent [Alfred Tarski].

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Order invariant definability

Motivation: Elements are in some arbitrary order in a computer.

Definition: A property is *order invariant* (in the finite) if it can be elementarily defined with an arbitrary linear order <, independent of that order.

- An ordered Boolean algebra has an *even number* of atoms if there is an element (set of atoms) containing the first atom in the linear order, alternate atoms thereafter, and not the last atom [Gurevich].

Example: If the atoms are $a_1 < a_2 < a_3 < a_4$, then $\{a_1, a_3\}$ is a witness.

- But, neither connectivity nor acyclicity are order invariant [Yuri].

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Motivation

- Order-invariance arises in logic because every structure input to a computer must be linearly ordered, in some arbitrary way.
- What if we were to also require the very natural condition that whenever possible, an input datum is *related* to previous data?
- In a simple graph, this type of ordering is called a *traversal*.
- There is an analogous notion of traversal-invariance.

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Graph traversals

Definition: A *traversal* of a connected simple graph is a linear ordering of its vertices in which every initial segment is connected.

OK: a — b — c — d <u>NOT</u>: a c — b d $\{a, c\}$

Note: Every node (except the first) must have a preceding neighbor.

- A *traversal* of an arbitrary simple graph is the direct ordered sum of traversals of its connected components: […] […] …

Fact: Every simple graph admits, e.g., a depth-first traversal.

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Traversal invariant definability – basic notion

Idea: A graph property P is *traversal invariant* if it can be defined by a first-order sentence θ on ordered graphs $(G, <)$ in which $<$ is a traversal of G. I.e., $(G, <) \models \theta \Leftrightarrow G \in P$ independent of the traversal $<$.

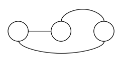
Examples: Let $(G, <)$ be a traversed graph

<u>Connected</u>: *every element except the first has a preceding neighbor*.

Why? The second component must start anew: […] […] …

<u>Acyclic</u>: *no element has two preceding neighbors.*

Why? The descending paths must eventually join; every cycle has an apex (maximal element).



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Traversal invariance – full version

• Close the notion of traversal invariance under interpretations.

$$S \to \pi(S) = G \to (G, <) \to \sigma(G, <)$$

1. A (finite) structure S in some signature;
2. A first-order translation π which results in a simple graph G;
3. A traversal $<$ of G; followed by
4. A traversal invariant sentence σ over simple ordered graphs.

Example: Bipartiteness (use the square of the edge relation)

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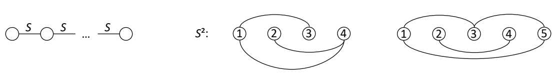
The importance of successor

• Without successor, we cannot determine the parity of the domain.

Known: capturing algorithms in logic requires a successor relation.

Example: An empty graph has even cardinality iff the square of its successor, with an edge between the endpoints, is connected.



Assume: All our graphs come with an arbitrary successor relation.

Main result: Traversal invariant FO logic with successor is logspace.

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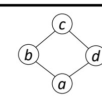
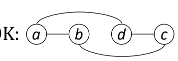
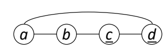
Breadth-first traversals



Definition: A graph traversal $(G, <)$ is *breadth-first* if the earliest preceding neighbor function p is monotone ($p(x) = x$ if undefined).

OK:  <u>NOT</u>:  $\because p(d) < p(c)$

Definition: A formula on ordered graphs is *breadth-first invariant* if it determines a query on a breadth-first traversal of the underlying simple graph independent of any particular breadth-first traversal.

Example: Are two nodes x and y equidistant from a fixed node c?

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Summary of results

Theorem: A problem on finite structures with successor is computable in (nondeterministic) logspace iff it is elementarily definable in a (breadth-first) traversal invariant manner. I.e.,

- L = FO(successor) + simple graph traversals
- NL = FO(successor) + breadth-first traversals

These characterizations augment FOL with a complete algorithm!

Contrast this with the usual way of providing a complete problem.

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The challenge of effectivity

• It is undecidable whether a formula is order-invariant in the finite.
Similarly, it is undecidable whether a formula is traversal-invariant.

• There is no *syntactic* way to effectively list all invariant formulas!

• But is there a *semantic* way to effectively list them?

1. Every formula on the list is traversal-invariant in the finite.
2. Every traversal-invariant formula is equivalent to one on the list.

Thank you! Any questions?

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