On the combinatorics of Girard's exponentials

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Structure Meets Power

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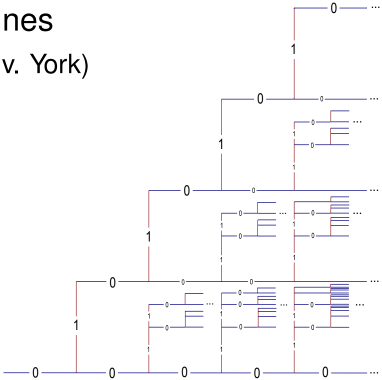


The topic of the talk

This is not a talk about logic!

(Linear, or otherwise)

It is about the **combinatorics** behind some logical models.

- Intuitive combinatorial interpretations of the structures used.
- Where we might find such structures in other settings.
- How such things generalise, from a *combinatorial* rather than a *logical* perspective.

It is nevertheless useful to have at least some idea of what is being modeled!

Our starting point :

The models we consider are of a fragment[1] of J.-Y. Girard's *Linear Logic*.

As first emphasised by Y. Lafont, this treats **formulæ** as **resources** that may be 'used up' in a deduction

$$\frac{A\ ,\ A \Rightarrow B}{B}$$

The resource A is *'consumed'* by Modus Ponens.

$$\frac{A\ ,\ A \Rightarrow B}{A\ ,\ B}$$

Resource A is still '*available for re-use*'.

This was a consequence of re-considering structural rules

These are rules to do with "how proofs are put together", rather than "how logical operators behave".

They nevertheless have consequences for logical operators, such as *commutativity* or *idempotency* of conjunction.

[1] Precisely, the multiplicative-exponential fragment.

This is the way the world ends? – restricting structural rules!

Unlike other substructural logics, LL does not entirely discard structural rules :

Affine logic rules out verious logical paradoxes (e.g. Kleene's paradox) by eliminating **contraction**.

Relevance logic keeps a 'causal link' between assumption and conclusion, by ruling out **weakening**.

Instead, these are *heavily controlled* by introducing two 'exponential' forms of each formula A

- $!(A)$ — **"of course"** or **"bang"**
- $?(A)$ — **"why not"** or **"whimper"**

that are susceptible to these rules, along with rules for introducing / manipulating them.

In particular $!(A)$ may be thought of as an **"infinitely re-usable version of A"**.

The precise setting of the talk

We consider the combinatorics of how “Of course” and “conjunction” are modeled, with particular reference to this concept of ‘infinite re-usability’.

This is within the setting of **GoI** :

- “Geometry of Interaction (I) : Interpretation of system $\mathcal{F}$”
  — J.-Y. Girard (1988)
- “Geometry of Interaction (II) : Deadlock-free algorithms
  — J.-Y. Girard (1988)

Both of these give representations of a fragment of linear logic.

The algebraic setting

Propositions are modeled as,

partial injective functions on the natural numbers

These are, equivalently :

1. Relations $f \subseteq \mathbb{N} \times \mathbb{N}$ satisfying,

$$a = a' \Leftrightarrow b = b' \text{ for all } (b', a'), (b, a) \in f$$

2. Partial functions that are bijections from their domain to their image.

These are

- closed under composition,
- include the identity, and all other bijections,
- closed under generalised inverse (relational converse)

and so form a monoid $\mathcal{I}(\mathbb{N})$ — the **symmetric inverse monoid** on $\mathbb{N}$.

The conjunction of partial injections

Given 'propositions' $f, g \in \mathcal{I}(\mathbb{N})$, their **conjunction** in the GoI system is :

$$[f \star g](n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even,} \\ 2.g\left(\frac{n-1}{2}\right) + 1 & n \text{ odd.} \end{cases}$$

Algebraically

An injective inverse monoid homomorphism

$$\star : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \to \mathcal{I}(\mathbb{N})$$

- $[f \star g][h \star k] = fh \star gk$.
- $[Id \star Id] = Id$

Categorically

A faithful symmetric semi-monoidal[a] tensor on an inverse monoid (i.e. single-object category).

[a] In the sense of Joyal & Kock's "weak units"

Thinking concretely

Alice and **Bob** play with an infinite deck of cards

1. A countably infinite deck of cards is dealt in the usual way to Alice and Bob.
2. Alice applies f to her hand of cards, and Bob applies g to his hand.
3. Alice and Bob's hands are then merged, using a perfect, interleaving, riffle shuffle.

Some subtleties :

- $\mathbb{N}$ has a bottom element, but no top element

  — cards are dealt from the bottom of the pack.

- f and g may be *partially defined*. For simplicity, we consider the *very special case* where they are bijections[2].

[2] As intuition for partiality, consider that Alice & Bob can erase the picture on a card, or insert blank cards ...

An unusual conjunction

Some relevant properties :

1. $[f \star g] \neq [g \star f]$
2. $f \star [g \star h]] \neq [[f \star g] \star h]$
3. $[f \star f] \neq f$

Some very standard category theory ...

Identities 1. and 2. hold, up to a fixed bijection. For all $f, g, h \in \mathcal{I}(\mathbb{N})$

$$\sigma[f \star g] = [g \star f]\sigma \quad , \quad \alpha[f \star [g \star h]] = [[f \star g] \star h]\alpha$$

Girard's conjunction is *semi-monoidal tensor* on the monoid $\mathcal{I}(\mathbb{N})$.

Identity 3. <u>cannot hold</u> in the same way.

Counterexample : Consider some f that is only defined for a single $n \in \mathbb{N}$. Observe that $[f \star f]$ is defined at both $2n$ and $2n + 1$.

Bringing in the bang!

Instead, we have $!(f) \in \mathcal{I}(\mathbb{N})$, which is the 'infinitary' form of f.

This satisfies the crucial fixed-point equation

$$f \star !(f) \; = \; !(f)$$

that allows it to be thought of as 'infinitely many copies of f'.

INFORMALLY $!(f) \; = \; f \star [f \star [f \star \ldots]]]$

FORMALLY $!(f) \; = \; \Phi(Id \times f)\Phi^{-1}$ where

$$\Phi(x, y) \; = \; 2^{x+1} y + 2^x - 1 \quad \forall \; (x, y) \in \mathbb{N} \times \mathbb{N}$$

is a bijection, monotone in both variables.

A few questions

1. In what setting can the informal description be made a formal limit?
2. Does it also relate to "shuffling decks of cards"?
3. What – if anything – is the significance of monotonicity w.r.t. the product order?

First – modeling shuffles

A (mathematical) strategy :

We take the (very well-studied) finite case, and, “check everything still works”.

Shuffles are modeled by **monotone bijections** :

Bijectivity ensures all cards are used,

Monotonicity accounts for,

“If card a is above card b before the shuffle,it is still above b afterwards.”

‘Multiple identical decks’ are given by disjoint unions,

$$\underbrace{\mathbb{N} \uplus \ldots \uplus \mathbb{N}}_{k \text{ times}} = \mathbb{N} \times \{0\} \cup \ldots \mathbb{N} \times \{k-1\} = \mathbb{N} \times \{0, \ldots k-1\}$$

These are ordered using the induced partial order :

$$(x, i) \leqslant (y, j) \quad \text{iff} \quad x \leqslant y \ \text{ and } \ i = j$$

A **deal** is then simply the inverse of a shuffle

Shuffles as Cantor points

As in the finite & infinite case, we may also describe a shuffle of k decks of cards

$$\Psi : \mathbb{N} \times \{0, \dots k-1\} \rightarrow \mathbb{N}$$

operationally, as a **sequence** $p_0, p_1, p_2, p_3, \dots$ over the set $\{0, \dots, k-1\}$.

This has the intuition of an operational description :

"Take from deck p_0, then p_1, then p_2, then ..."

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\Psi^{-1}"] \arrow[dr, "seq_\Psi"'] & \mathbb{N} \times \{0, \dots k-1\} \arrow[d, "\pi_2"] \\
& \{0, \dots, k-1\}
\end{tikzcd}$$

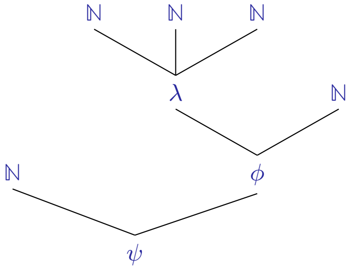
This is the **sequence of plays** for Ψ, a point of the Cantor space $\mathcal{C}_{\{0,\dots,k-1\}}$ over the set $\{0, \dots, k-1\}$. It is enough to characterise Ψ, by monotonicity & bijectivity.

Operads of card shuffles

Unsurprisingly, plugging together card shuffles forms an (non-symmetric) operad.

(It is an example of a standard construction :
the endomorphism operad in a semi-monoidal category)

A tree such as :



represents a shuffle (i.e. monotone bijection) of five decks of cards :

$$\psi(1_{\mathbb{N}} \uplus \phi(\lambda \uplus 1_{\mathbb{N}})) \ : \ \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \ \rightarrow \ \mathbb{N}$$

We define the k-deck **riffle shuffle** $\Omega_k : \mathbb{N}^{\uplus k} \to \mathbb{N}$ to be the bijection $\Omega_k(n,i) = kn+i$ for all $(n,i) \in \mathbb{N}^{\uplus k}$, with the natural diagrammatics :

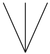
$\Omega_1 : \quad \mathbb{N} \to \mathbb{N}$

$n \mapsto n$

$\Omega_2 : \quad \mathbb{N} \uplus \mathbb{N} \to \mathbb{N}$

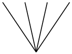
$(n,0) \mapsto 2n$
$(n,1) \mapsto 2n+1$

$\Omega_3 : \quad \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \to \mathbb{N}$



$(n,0) \mapsto 3n$
$(n,1) \mapsto 3n+1$
$(n,2) \mapsto 3n+2$

$\Omega_4 : \quad \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \uplus \mathbb{N} \to \mathbb{N}$



$(n,0) \mapsto 4n$
$(n,1) \mapsto 4n+1$
$(n,2) \mapsto 4n+2$
$(n,3) \mapsto 4n+3$

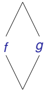
The inverse of Ω_n is the n-player **fair deal**.

Shuffles & bangs in conjunction

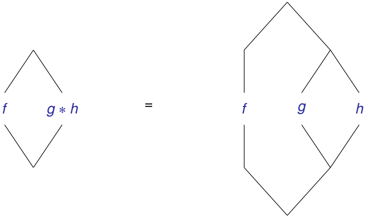
Girard's conjunction is based on the binary case :

The two-player fair deal $\Omega_2^{-1} : \mathbb{N} \to \mathbb{N} \uplus \mathbb{N}$

The two-pack riffle shuffle $\Omega_2 : \mathbb{N} \uplus \mathbb{N} \to \mathbb{N}$

We draw his conjunction in the natural way as $[f \star g] =$ 

and interpret bracketing

as tree structure, so $[f \star [g \star h]]$ is given by 

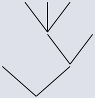
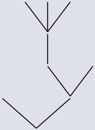
We nevertheless consider the general setting, and treat his conjunction as a special case . . . without worrying too much about logical interpretations!

The object of study

We define $\mathcal{R}iff$, the operad of **hierarchical riffle shuffles** to be the operad generated by the perfect riffle shuffles $\{\Omega_k\}_{k>1}$

The obvious diagrammatics :

- As we only have one generator of each arity, we may draw H-R shuffles as *unlabled planar trees*.
- We leave identities implicit.

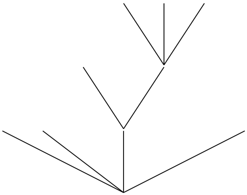
We do not distinguish between  and 

Claim : Each k-leaf tree determines a <u>distinct</u> monotone Hilbert-hotel style bijection from k copies of $\mathbb{N}$ to a single copy of $\mathbb{N}$, so $\mathcal{R}iff$ is isomorphic to the formal operad *RPT* of *rooted planar trees*[3].

[3] Some fun may be had labeling facets of associahedra by elements of $\mathcal{R}iff$. That is the subject of another talk . . .

What bijections do they determine??

An illustrative example : $\mathcal{T} \;=\; \Omega_4 \circ_3 (\Omega_2 \circ_2 \Omega_3) \;=\; (\Omega_4 \circ_3 \Omega_2) \circ_4 \Omega_3$



$$\mathcal{T}(n,i) = \begin{cases} 4n & i=0 \\ 4n+1 & i=1 \\ 8n+2 & i=2 \\ 24n+6 & i=3 \\ 24n+14 & i=4 \\ 24n+22 & i=5 \\ 4n+3 & i=6 \end{cases}$$

Each $\mathcal{T}(\_,i)$ is a linear injection $n \mapsto X_i n + Y_i$.

As $\mathcal{T}$ is a *bijection*, $im(\mathcal{T}(\_,i)) \;\cap\; im(\mathcal{T}(\_,j)) \;=\; \varnothing$ and $\bigcup_{i=0}^{7} im(\mathcal{T}(\_,j)) \;=\; \mathbb{N}$

Every member of $\mathcal{R}iff$ '**covers the natural numbers with linear sequences**'

They determine ~~distinct~~ finite open covers of $\mathbb{N}$ w.r.t. the profinite topology.

From riffle shuffles to congruential bijections

Consider $S, T \in \mathcal{R}iff_k$ where

$$T(n,i) = X_i n + Y_i \quad \text{and} \quad S(n,i) = A_i n + B_i$$

What happens when we

1. deal out a deck of cards using S^{-1}
2. then shuffle them back together using T?

We derive a piecewise-linear bijection on $\mathbb{N}$,

$$TS^{-1}(n) \;=\; X_i \left(\frac{n - B_i}{A_i} \right) + Y_i \quad \text{where} \quad n \ (mod\ A_i) = B_i$$

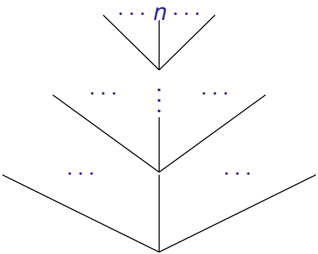
These are *congruential functions* introduced by John Conway ("Unpredictable iterations" 1971), to prove formal undecidability of iterative problems such as Collatz's $3x + 1$ problem.

(A result heavily prefigured in Sergei Maslov's "On E. L. Post's Tag Problem" 1964)

Counting coefficients

How do we compute such (indexed families) of linear maps?

The general case : card n from deck i :



Branch a_k out of b_k

$\cdots$

Branch a_2 out of b_2

Branch a_1 out of b_1

We have an injection $n \mapsto X_i n + Y_i$. How to compute X_i and Y_i ?

Multiplicative coefficients Trivially, $X_i = \prod_{j=1}^{k} b_j$.

Additive coefficients We can simply <u>write down</u> the value of Y_i.

Relating two strands of Cantor's work

Über Einfache Zahlensysteme – G. Cantor (1869)

On Simple Number Systems studied **mixed-radix** counting : positional number systems where the *base* used varies between *columns*.

Familiar example : pre-decimal / post-brexit British currency

4 Farthings = 1 Penny , 12 Pennies = 1 Shilling , 20 Shillings = 1 Pound ...

We may simply write down the value of Y_i

$$Y_i \ = \ \begin{array}{|c|c|c|c|} \hline \text{base } b_k & \text{base } b_{k-1} & \ldots & \text{base } b_1 \\ \hline a_k & a_{k-1} & \ldots & a_1 \\ \hline \end{array}$$

(Note : $b_k b_{k-1} \ldots b_1$ is an ordered factorisation of X_i).

Transformations between different mixed-radix counting systems are particularly well-studied in the Fast Fourier Transforms re-discovered by Cooley & Tukey (... but originally due to Gauss).

To justify the claim of "uniqueness"

Proposition : $\mathcal{R}iff$ is *freely generated* by $\{\Omega_j\}_{j=2,3,4,\ldots}$.

No two distinct k-leaf trees determine the same bijection from $\mathbb{N}^{\uplus k}$ to $\mathbb{N}$.
i.e. $\mathcal{R}iff$ is isomorphic to the formal operad **rpt** of "rooted planar trees".

Proof (outline) : A simple induction argument on the number of leaves.

The only non-trivial step

We need to show that the generating set $\{\Omega_K\}_{k>0}$ is *minimal* — no perfect riffle can be produced by composing other perfect riffles.

We do this by showing that the generators Ω_K are a very special type of shuffle.

Definition A shuffle of k decks of cards $\Psi : \mathbb{N} \times \{0, \ldots, k-1\} \to \mathbb{N}$ is **standard** when it is **monotone in both variables**.

Standard shuffles

An operational characterisation :

This has the natural interpretation that, at any stage of the shuffle,

$$\begin{array}{c}\#\text{ of cards placed from deck } i \\ \geqslant \\ \#\text{ of cards placed from deck } i+1\end{array}$$

As a consequence, the sequence of plays will be an infinitary Ballot sequence.

Equivalently : The tableau determined by a standard shuffle is a (infinitary) standard Young tableau, with ordered rows & columns.

$\Psi(0,0)$	$\Psi(1,0)$	$\Psi(2,0)$	$\Psi(3,0)$	$\ldots$
$\Psi(0,1)$	$\Psi(1,1)$	$\Psi(2,1)$	$\Psi(3,1)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\Psi(0,k-1)$	$\Psi(1,k-1)$	$\Psi(2,k-1)$	$\Psi(3,k-1)$	$\ldots$

The generators $\{\Omega_1, \Omega_2, \Omega_3, \ldots\}$ are certainly standard – which composites are similarly standard?

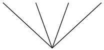
Characterising standard riffle shuffles

For a composite $S \circ_k T$ to be standard, we need the following :

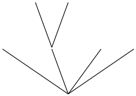
1. S and T are themselves both standard.
2. S is of arity k – i.e. the product is the (associative) **overproduct** $S \urcorner T \overset{def.}{=} S \circ_k T \ \forall S \in \mathcal{R}iff_k$ given by **grafting onto the far right leaf**.

As an illustrative example, consider $\Omega_4 \circ_2 \Omega_2$. All the generators are standard, but this composite is not standard :

0	4	8	12	16	...
1	5	9	13	17	...
2	6	10	14	18	...
3	7	11	15	19	...



0	4	8	12	16	...
1	9	17	25	33	...
5	13	21	29	36	...
2	6	10	14	18	...
3	7	11	15	19	...



The ‘standard’ property is preserved precisely when the final row is split.

Standard $\equiv$ Right-Associated

We may characterise standard hierarchical riffle shuffles

These are given by arbitrary (finite) overproducts of generators.

$$\Omega_{x_0} \urcorner\!\downarrow \Omega_{x_1} \urcorner\!\downarrow \Omega_{x_2} \urcorner\!\downarrow \ldots \urcorner\!\downarrow \Omega_{x_N}$$

No generator is a non-trivial composite of this form; therefore, the generating set is minimal, and by induction $\mathcal{R}iff$ is freely generated.

Every distinct finite sequence of natural numbers determines a distinct standard shuffle / standard Young tableau, by

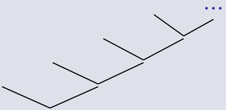
$$n_0 n_1 \ldots n_x \mapsto \Omega_{n_0+2} \urcorner\!\downarrow \Omega_{n_1+2} \urcorner\!\downarrow \ldots \urcorner\!\downarrow \Omega_{n_x+2}$$

i.e. there exists an injective monoid homomorphism from the free monoid over the natural numbers to $(\mathcal{R}iff, \urcorner\!\downarrow)$, given by $std(n) \stackrel{def}{=} \Omega_{n+2}$.

Taking things to the limit ...

Can we give meaning to **infinitary** overproducts??

In particular, for the 2-player riffle shuffle / fair deal, can we give a meaning to :



Doing so would allow us to formalise the bang as an ‘infinitary conjunction’

$$!(f) \;=\; [f \star [f \star [f \star [f \star \ldots]]]]$$

Precisely, is the limit $\Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \ldots$ well-defined?

If so, how about other infinitary overproducts of generators??

From monoids to Cantor spaces

We may extend our monoid homomorphism to *one-sided infinite* strings (i.e. points of $\mathcal{C}_{\mathbb{N}}$, the Cantor space over the natural numbers) in a natural way.

Consider some infinite sequence $\Omega_{x_0} \urcorner\!\!\downarrow \Omega_{x_1} \urcorner\!\!\downarrow \Omega_{x_2} \urcorner\!\!\downarrow \Omega_{x_3} \urcorner\!\!\downarrow \ldots$ along with the sequence of tableaux determined by the prefixes :

1. Ω_{x_0}
2. $\Omega_{x_0} \urcorner\!\!\downarrow \Omega_{x_1}$
3. $\Omega_{x_0} \urcorner\!\!\downarrow \Omega_{x_1} \urcorner\!\!\downarrow \Omega_{x_2}$
4. $\Omega_{x_0} \urcorner\!\!\downarrow \Omega_{x_1} \urcorner\!\!\downarrow \Omega_{x_2} \urcorner\!\!\downarrow \Omega_{x_3}$

At each step, every natural number N either:

- moves left (& possibly downwards as well), or
- stays in the same place ... at which point it remains there!

$$\forall (N,j) \in \mathbb{N} \times \mathbb{N},\ \exists K \in \mathbb{N}\ s.t.\ \forall T, T' \in \mathcal{R}iff$$
$$(\Omega_0 \urcorner\!\!\downarrow \ldots \Omega_K \urcorner\!\!\downarrow T)(N,j) = (\Omega_0 \urcorner\!\!\downarrow \ldots \Omega_K \urcorner\!\!\downarrow T')(N,j)$$

These will define *infinitary standard shuffles*, or monotone bijections $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$.

The simplest worked example:

The simplest is the infinitary overproduct $\Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \ldots$ that may be thought of as “the right fixed point for the binary riffle shuffle”

$$\Omega_2 \rdsh (\Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \ldots) \quad = \quad \Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \Omega_2 \rdsh \ldots$$

Using the ‘counting branches’ method for determining coefficients leads us to Girard’s bijection $\Phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

$$\Phi(x, y) \;=\; 2^{x+1} y + 2^x - 1$$

Its description as an overproduct accounts for monotonicity in *both* variables.

A simple question

There are two different ways of describing a (standard) shuffle :

- A (monotone in both variables) *bijection*.
- The corresponding (ballot) *sequence of plays*.

What do both of these look like??

For people who prefer *finite* Young tableaux ..

Giving the tableaux for Φ explicitly :

0	2	4	6	8	10	$\dots$
1	5	9	13	17	21	$\dots$
3	11	19	27	35	43	$\dots$
7	23	39	55	71	87	$\dots$
15	47	79	111	143	175	$\dots$
31	94	159	223	287	351	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

As a general pattern

The sub-tableaux given by considering the first n natural numbers

form an inclusion-ordered unbounded sequence of

finitary standard Young tableaux,

for **any** monotone bijection

$$\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

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As a <u>general pattern</u>

The sub-tableaux given by considering the first n natural numbers

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finitary standard Young tableaux,

for **any** monotone bijection

$$\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

For people who prefer *finite* Young tableaux ..

Giving the tableaux for Φ explicitly :

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3	11	19	27	35	43	...
7	23	39	55	71	87	...
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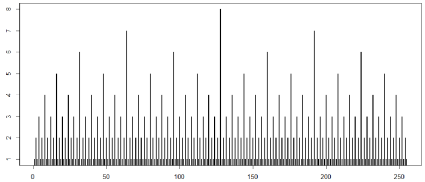
From bijections to sequences

Alternatively, the **sequence of plays** $\pi_2 \Phi^{-1} : \mathbb{N} \to \mathbb{N}$ is given by

0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4	0	1
0	2	0	1	0	3	0	1	0	2	0	1	0	5	0	1	0	2
0	1	0	3	0	1	0	2	0	...								

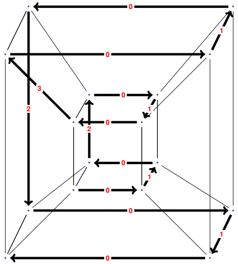
This is the (ballot) **ruler sequence** — sequence number A007814 in the Online Encyclopedia of Integer Sequences (`https://oeis.org/A007814`), where it is characterised as $r(n) + 1 =$ *"The Hamming distance between n and $n+1$"*



Picture taken from "On the ubiquity of the Ruler sequence" – J. Nuño, F. Muñoz (2020)

A fun application

Among *many other applications*, the ruler sequence $r(n)$ is known in network topology for determining Hamiltonian paths in hypercube graphs :



The simple prescription :

- Index axes (i.e. dimensions) by the natural numbers,
- On step n, move along axis $r(n)$.

visits each vertex exactly once.

Concretely, *how could we perform this shuffle??*

The ruler series is the sequence of plays for the bijection $\Phi = \Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \Omega_2 \urcorner\!\!\downarrow \ldots$

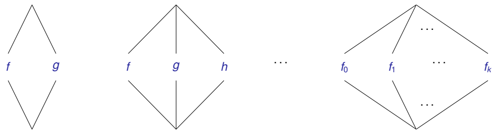
				1	**0**
			1	0	**1**
			1	1	**0**
		1	0	0	**2**
		1	0	1	**0**
		1	1	0	**1**
		1	1	1	**0**
	1	0	0	0	**3**
	1	0	0	1	**0**
	1	0	1	0	**1**
	1	0	1	1	**0**
	1	1	0	0	**2**
	1	1	0	1	**0**
	1	1	1	0	**1**
	1	1	1	1	**0**
1	0	0	0	0	**4**

For an arbitrary (infinite) overproduct $\Omega_{x_0} \urcorner\!\!\downarrow \Omega_{x_1} \urcorner\!\!\downarrow \Omega_{x_2} \urcorner\!\!\downarrow \Omega_{x_3} \urcorner\!\!\downarrow \ldots$, we simply count in a mixed-radix system with columns labeled by $\ldots, x_3, x_2, x_1, x_0$.

Question : Is there some setting for which such infinite overproducts provide an infinitary form of conjunction?

Generalising conjunctions

Girard's conjunction naturally generalises to an $\mathbb{N}$-indexed family of injective homomorphisms, given by conjugation by Ω_k, for all $k > 1$.



The intuition :

A pack of cards is dealt out amongst k players, using a fair deal. Each player j then applies f_j to his hand of cards. All hands of cards are then shuffled together using the perfect riffle shuffle Ω_K.

Writing this out explicitly,

$$[f_0 \star f_1 \star \ldots \star f_{k-1}](n) \;=\; k.f_r\left(\frac{n-r}{k}\right) + r \;\; \textit{where} \;\; n \; (\textit{mod } k) = r$$

Each of these defines an *injective inverse monoid homomorphism* $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$

Generalised conjunctions as an operad

Define $\mathcal{BOB}$, the operad of **Bobzien Conjunctions** to be generated by

$$\begin{aligned}
[_ \star _] &: \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \to \mathcal{I}(\mathbb{N}) \\
[_ \star _ \star _] &: \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \to \mathcal{I}(\mathbb{N}) \\
[_ \star _ \star _ \star _] &: \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \to \mathcal{I}(\mathbb{N}) \\
&\vdots
\end{aligned}$$

It is a sub-operad of the endomorphism operad of $\mathcal{I}(\mathbb{N})$ in the monoidal category $(\mathbf{Inv}, \times)$ of inverse monoids with Cartesian product.

Note : it is *freely generated* by one generator of each arity and hence also isomorphic to the operad *RPT* or rooted planar trees.

Why “Bobzien conjunctions” ?

“The Combinatorics of Stoic Conjunction”

— S. Bobzien (2011) *Oxford Studies in Ancient Philosophy*

This analyses the somewhat mysterious statement in Plutarch’s **Quaestiones Convivales** :

> *“*Chrysippus *says that the number of conjunctions [constructible] from only ten assertibles exceeds one million. However,* Hipparchus *refuted this, demonstrating that the affirmative encompasses* **103049** *conjoined assertibles.”*

As pointed out by Daniel Hough (c. 1994), this is the 10^{th} *little Schröder number* — the number of distinct 10-leaf rooted planar trees.

Suzanne Bobzien analysed the logical assumptions Hipparchus & Chrysippus must have made, to arrive at their figures.

Some logical(?) assumptions!

"Hipparchus counts the same sequence of conjuncts but with different bracketing as different conjunctions. He counts

$$p \wedge [q \wedge r] \ , \ [p \wedge q \wedge r] \ , \ [[p \wedge q] \wedge r]$$

as different assertibles. " – *S.B.*

"In order to get to this number, Hipparchus took the order of atomic assertibles as fixed, so $[p \wedge q] \neq [q \wedge p]$" – *S.B.*

"Non-simple [assertibles] are those that are put together from an assertible that is <u>taken twice</u>, or from different assertibles." – *Plutarch, quoted by S.B.*

Isomorphic $\neq$ Identical

The Bobzien conjunctions

$$[f \star g \star h] \ , \ [[f \star g] \star h] \ , \ [f \star [g \star h]]$$

are nevertheless *equivalent up to (fixed) isomorphism*.

$$\begin{tikzcd}
& {[f \star g \star h]} \arrow[dl, "\gamma_L(_)\gamma_L^{-1}"'] \arrow[dr, "\gamma_R(_)\gamma_R^{-1}"] & \\
{[[f \star g] \star h]} & & {[f \star [g \star h]]} \arrow[ll, "\alpha(_)\alpha^{-1}"]
\end{tikzcd}$$

The congruential bijection $\alpha : \mathbb{N} \to \mathbb{N}$ is the canonical associativity isomorphism for Girard's conjunction, considered as a categorical tensor. What about the other two congruential bijections??

A precursor to a famous problem

$$\alpha(n) = \begin{cases} 2n & n \ (mod\ 2) = 0 \\ n+1 & n \ (mod\ 4) = 1 \\ \frac{n-1}{2} & n \ (mod\ 4) = 3 \end{cases}$$

The **associator**

$$\gamma_L(n) = \begin{cases} \frac{4n}{3} & n \ (mod\ 3) = 0 \\ \frac{4n+2}{3} & n \ (mod\ 3) = 1 \\ \frac{2n-1}{3} & n \ (mod\ 3) = 2 \end{cases}$$

The **(left) Collatz bijection**

$$\gamma_R(n) = \begin{cases} \frac{2n}{3} & n \ (mod\ 3) = 0 \\ \frac{4n-1}{3} & n \ (mod\ 3) = 1 \\ \frac{4n+1}{3} & n \ (mod\ 3) = 2 \end{cases}$$

The **(right) Collatz bijection**

The $3x + 1$ problem & its generalisations – Jeffrey Lagarias (1985)

Writing about L. Collatz : *"In his notebook dated July 1, 1932, he considered the function*

$$n \mapsto \begin{cases} \frac{2}{3}n & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{4}{3}n - \frac{1}{3} & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{4}{3}n + \frac{1}{3} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

He posed the problem of whether the cycle containing 8 *is finite or infinite. I will call this the* **Original Collatz Problem**. *His original question has never been answered."*

The nature of my game

Alice and Bob play a game against a dealer, with a countably infinite pack of cards. The Dealer deals out this pack to all players, using a fair deal.

- Alice and Bob merge their stacks together, using a perfect riffle shuffle.
- The Dealer merges the result of this with his stack, again using a perfect riffle.

The process repeats. Each round of the game permutes the infinite pack of cards

Alice and Bob will win, and may leave the game, when

one card, that they mark beforehand,

returns to its original position in the Dealer's hand.