An overview of weak distributive laws

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SmP, 28 June 2021

Joint work with Daniela Petrişan and Marc Aiguier

Distributive laws

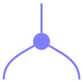
Monads = effects

effect   create effect   collapse effects

- nondeterminism $= x \vee y$
- probability $= x \oplus_p y$

Distributive laws

Monads = effects



effect create effect collapse effects

- nondeterminism $= x \vee y$
- probability $= x \oplus_p y$

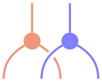
Distributive laws = compositionality of effects

+ 4 axioms:

create
collapse
create
collapse

swap effects

The composite effect is



effect create effect collapse effects

Weak distributive laws

No-go theorems. Composing

- $\vee$ and $\vee$ is impossible
- $\vee$ and $\oplus_p$ is impossible
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Forcing distributivity

$$x \oplus_p (y \vee z) = (x \oplus_p y) \vee (x \oplus_p z)$$

induces a weak distributive law (in Garner's sense) between $\vee$ and $\oplus_p$

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swap effects

$+$ 3 axioms:

$\times$

collapse

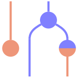
create

collapse

and a weak composite effect

Examples

Trivial weak distributive laws

Any  induces a trivial weak distributive law 

i.e. the weak composite effect is just the blue effect.

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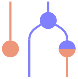
Any  induces a trivial weak distributive law 

i.e. the weak composite effect is just the blue effect.

Monotone weak distributive laws include, in the category of sets

- $\vee$ and $\vee$
- $\vee$ and $\oplus_p$
- $\vee$ and $+$ where $+$ is semiring choice
- $\vee$ and $\rightarrow$ where $\rightarrow$ is ultrafilter convergence

and in other categories

- $\vee$ and $\vee$ in any topos (graphs, nominal sets...)
- $\vee$ and $\vee$ in compact Hausdorff spaces

Applications

Compositionality

- composite algebraic theories e.g. $x \oplus_p (y \vee z) = (x \oplus_p y) \vee (x \oplus_p z)$
- iterated weak distributive laws to combine more than 2 effects

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Coalgebra

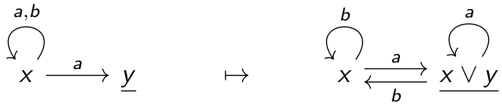
- monads = effects = branching types for state-based systems
- generalised powerset construction for automata ✓
- up-to techniques for bisimulations

Generalised powerset construction

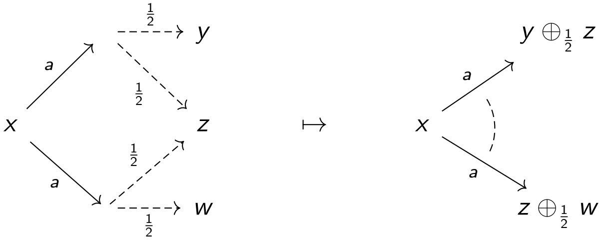
$$\begin{tikzcd} x \arrow[loop above, "{a,b}"] \arrow[r, "a"] & \underline{y} & \mapsto & x \arrow[loop above, "b"] \arrow[r, shift left, "a"] & \underline{x \vee y} \arrow[l, shift left, "b"] \arrow[loop above, "a"] \end{tikzcd}$$

- Powerset construction relies on a (monad-functor) distributive law.

Generalised powerset construction



- Powerset construction relies on a (monad-functor) distributive law.



On the right, x can a-transition to any distribution $(y \oplus_{\frac{1}{2}} z) \oplus_p (z \oplus_{\frac{1}{2}} w)$.

- Generalised powerset construction relies on a weak distributive law.

Conclusion

Main open question: continuous probability and nondeterminism

- $\vee$ and $\vee$ in compact Hausdorff spaces ✓
- $\vee$ and $\oplus_p$ in compact Hausdorff spaces ?

  $\vee$ = Vietoris monad and $\oplus_p$ = Radon monad

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Other future work

- Find non-trivial non-monotone weak distributive laws, e.g.

  $\oplus_p$ and $\vee$

Conclusion

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Other future work

- Find non-trivial non-monotone weak distributive laws, e.g.

  $\oplus_p$ and $\vee$
- Coweak distributive laws = dual framework, much less well-behaved

swap effects

+ 3 axioms: create collapse $\times$ collapse

Thank you!

Any questions?

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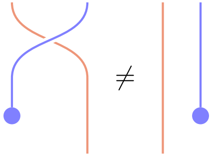
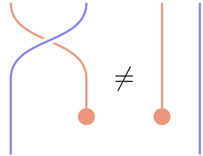
Failing diagrams

Weak distributive laws:



Coweak distributive laws:



The slides quote [11, 2, 10, 13, 15, 14, 8, 7, 4, 9, 1, 6, 3, 12, 5]