Compositionality and Logical Definability for Monads

Achim Blumensath

Composition results in logic

Typical example: disjoint unions

$$\mathfrak{A} \equiv_{\mathrm{MSO}_m} \mathfrak{A}' \quad \mathfrak{B} \equiv_{\mathrm{MSO}_m} \mathfrak{B}' \quad \Rightarrow \quad \mathfrak{A} \oplus \mathfrak{B} \equiv_{\mathrm{MSO}_m} \mathfrak{A}' \oplus \mathfrak{B}'$$

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Composition Theorem (weak version)

$$\mathfrak{A}_i \equiv_L \mathfrak{B}_i \quad \Rightarrow \quad F(\mathfrak{A}_i)_i \equiv_L F(\mathfrak{B}_i)_i$$

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Composition Theorem (strong version)

$$F(\mathfrak{A}_i)_{i \in I} \vDash \varphi \quad \Leftrightarrow \quad I^+ \vDash \psi$$

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Case study: monads

- class $\mathbb{M}A$ of 'A-labelled structures'
- composition operation $\mathrm{flat} : \mathbb{M}\mathbb{M}A \to \mathbb{M}A$

  (usually assumed to be associative, turning $\mathbb{M}$ into a **monad**)

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finite/infinite trees, countable linear orders, series-parallel graphs,…

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Example $\mathbb{M}A = A^*$ with concatenation

- $\mathbb{M}A$ is the free monoid over A.
- $\langle A, \pi \rangle$ is just a monoid with

  $\pi(\langle a_0, \ldots, a_{n-1} \rangle) = a_0 \cdots a_{n-1}$

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Recognisability

$K = \varphi^{-1}[P]$ for $\varphi : \mathbb{M}X \to \mathfrak{A}$ and $P \subseteq A$

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Example $\mathbb{M}X = X^*$

$K \subseteq X^*$ is regular iff it is recognised by a finite monoid.

Logic

Logic $\langle L, \mathcal{M}, \models \rangle$

- formulae L
- models $\mathcal{M}$
- satisfaction relation $\models \subseteq \mathcal{M} \times L$

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$L = \mathrm{MSO}[\Sigma]$ and $\mathcal{M} = \mathrm{STR}[\Sigma]$

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Morphisms

$\langle \lambda, \mu \rangle : \langle L, \mathcal{M}, \models \rangle \to \langle L', \mathcal{M}', \models' \rangle$

where $\lambda : L \to L'$ and $\mu : \mathcal{M}' \to \mathcal{M}$ such that

$$\mu(\mathfrak{M}') \models \varphi \quad \text{iff} \quad \mathfrak{M}' \models \lambda(\varphi)$$

Example interpretations

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Logical equivalence

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$\mathbb{M}$-Compositionality (for $\mathcal{M} = \mathbb{M}X$)

$$s \sqsubseteq_L^{\mathbb{M}} t \quad \Rightarrow \quad \mathrm{flat}(s) \sqsubseteq_L \mathrm{flat}(t)$$

where $s \sqsubseteq_L^{\mathbb{M}} t$ means that $s(v) \sqsubseteq_L t(v)$ for all positions v

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Theory algebra

$\Theta_L X := \mathbb{M}X/\sqsubseteq_L$ if L is $\mathbb{M}$-compositional

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If $\mathrm{Syn}(K)$ exists, it is the minimal algebra recognising K.

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Example (Schützenberger)

A class $K \subseteq \Sigma^*$ is FO-definable if, and only if, $\mathrm{Syn}(K)$ satisfies

$x^n = x^{n+1}$ for some n.

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Requirements

- $\mathbb{M}$ sufficiently 'nice' (e.g., polynomial functor).
- L is $\mathbb{M}$-compositional.
- Every $K \in \mathcal{D}$ class has a finitary syntactic algebra $\mathrm{Syn}(K)$.
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Application

These conditions are satisfied for MSO_m and FO_m over infinite trees.

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- If K is L-definable then $\mathrm{Syn}(K)$ is L-definable.

  (If L is sufficiently nice.)