Quantifiers and Measures, Part II

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Taking over where Luca stopped...

The image of

$$R_f \colon \beta(\mathrm{Mod}_{n-1}) \to \mathcal{V}(\mathrm{Typ}_n)$$

is the Stone dual of $B_{\exists x_n} = \langle \exists x_n \varphi \mid \varphi \in \mathrm{FO}_n \rangle$.

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is the Stone dual of $B_{\exists x_n} = \langle \exists x_n \varphi \mid \varphi \in \mathrm{FO}_n \rangle$.

The construction $B \rightsquigarrow B_{\exists x_n}$ works for any

$$B \hookrightarrow \mathcal{P}(\mathrm{Mod}_n) \quad \text{dually given by} \quad f \colon \beta(\mathrm{Mod}_n) \twoheadrightarrow X$$

then we build

$$R_f \colon \beta(\mathrm{Mod}_{n-1}) \to \mathcal{V}(X).$$

And $B_{\exists x_n}$ can be identified with a subalgebra of $\mathcal{P}(\mathrm{Mod}_n)$.

The Boolean algebra of formulas

Inductively,

$$
\begin{aligned}
B_0^{(n)} &= \mathrm{QF}(x_1, \ldots, x_n) \\
B_{i+1}^{(n)} &= \text{the image of} \quad B_{\exists x_1}^{(n)} + \ldots + B_{\exists x_n}^{(n)} + B_i^{(n)} \to \mathcal{P}(\mathrm{Mod}_n)
\end{aligned}
$$

we build

$$
FO \;=\; \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_i^{(n)}
$$

as a Boolean subalgebra of $\mathcal{P}(\mathrm{Mod}_\omega)$.

Inductive constructions in domain theory

In DTLF operators $+$, $\times$, $\mathcal{P}_P$, $\mathcal{P}_H$, $\mathcal{P}_S$, $\rightarrow$ on the space side dually correspond to enrichments of logic.

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The solution of a domain equation

$$D \cong \sigma(D)$$

computed as a bilimit, dually adds logical connectives, step by step.

Vietoris as a space of measures

Closed subsets of a Stone space X

$$\updownarrow$$

finitely additive measures on $X \to \mathbf{2}$

(where $\mathbf{2} = (\{0,1\}, \wedge, \vee, 0, 1)$)

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functions $\mu \colon \mathsf{Clp}(X) \to \mathbf{2}$ s.t.

- $\mu(\emptyset) = 0$
- $A \cap B = \emptyset \implies \mu(A \cup B) = \mu(A) \vee \mu(B)$

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Via the correspondence

$$C \mapsto \mu_C \quad \text{such that} \quad \mu_C(A) = \begin{cases} 1 & C \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This yields a homeomorphism $\mathcal{V}(X) \cong \mathcal{M}(X, \mathbf{2})$.

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$$A \models \exists_k x.\varphi(x) \quad \text{iff} \quad \underbrace{1 + \cdots + 1}_{\text{for every } a \in A \text{ s.t. } A \models \varphi(a)} = k \ \text{ in } S$$

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- more... ?

Stone pairings [Nešetřil, Ossona de Mendez, 2013]

For a formula $\varphi(x_1, \ldots, x_n)$ and a finite structure A,

$$\langle \varphi, A \rangle \quad = \quad \frac{|\{\ \overline{a} \in A^n \mid A \models \varphi(\overline{a})\ \}|}{|A|^n} \qquad \textit{(Stone pairing)}$$

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Mapping $A \mapsto \langle -, A \rangle$ defines an embedding

$$\mathrm{Fin} \hookrightarrow \mathcal{M}(\mathrm{Typ}, [0,1])$$

Recall that Typ is dual to FO, i.e. clopens are of the form $[\![\varphi]\!]$ for $\varphi \in \mathrm{FO}$.

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which lifts uniquely to

$$\beta(\mathrm{Fin}) \to \mathcal{M}(\mathrm{Typ}, [0,1])$$

Motivation:
The limit of $(A_i)_i$ is computed as $\lim_{i \to \infty} \langle -, A \rangle$ in $\mathcal{M}(\mathrm{Typ}, [0,1])$.

The dual space of the image?

What is the dual of X?

$$\beta(\mathrm{Fin}) \twoheadrightarrow X \hookrightarrow \mathcal{M}(\mathrm{Typ}, [0,1])$$

$\mathcal{M}(\mathrm{Typ}, [0,1])$ has no non-trivial clopen! $\implies \mathsf{Clp}(X) \cong \mathbf{2}$

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Two possible solutions:

1. Describe X in terms of geometric logic, logic of proximity lattices or de Vries algebras, ...
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Two possible solutions:

1. Describe X in terms of geometric logic, logic of proximity lattices or de Vries algebras, ...
2. Replace $[0, 1]$ to retain classical logic.
   $\rightsquigarrow$ Our choice today!

The Stone space $\mathbf{\Gamma}$ (motivation)

Problem: We need to replace [0,1] by a similar space $\mathbf{\Gamma}$ s.t.

1. we can define measures $X \to \mathbf{\Gamma}$
2. the space $\mathcal{M}(X, \mathbf{\Gamma})$ is compact 0-dimensional
3. Stone pairing $\langle -, - \rangle : \mathrm{Fin} \to \mathcal{M}(\mathrm{Typ}, \mathbf{\Gamma})$ definable and is "comparable" with the original Stone pairing

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Observe: For $\varphi(v_1, \ldots, v_k)$, the Stone pairing $\langle \varphi, A \rangle$ takes values in a *finite* chain

$$I_n \;=\; \left(0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right)$$

where $n = |A|^k$.

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where $n = |A|^k$.

$\Longrightarrow$ Define $\mathbf{\Gamma}$ as an inverse limit of those (discrete) posets!

The Stone space Γ (description)

Define

$$\mathbf{\Gamma} = \lim\{f^n_{nm}\colon I_{nm} \twoheadrightarrow I_n\}_{n,m\in\mathbb{N}} \quad \text{where} \quad f^n_{nm}(\frac{a}{nm}) = \frac{\lfloor a/m \rfloor}{n}.$$

Elements of $\mathbf{\Gamma}$ are vectors

$$(x_n)_n \in \prod_n I_n$$

such that $f^n_{nm}(x_{nm}) = x_n$, for every $n, m \in \mathbb{N}$.

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Intuitively: coordinates represent approximations of numbers in [0,1] from bottom. The larger the n the better the approximation.

This gives $\begin{cases} \text{one representation of irrational numbers: } r^- \\ \text{two representations of rational numbers: } q^-,\ q^\circ \end{cases}$

$\mathbf{\Gamma}$ =

Properties of $\mathbf{\Gamma}$

- The subspace topology $\mathbf{\Gamma} \subseteq \prod_n I_n$ is compact 0-dimensional
- Retraction-section maps $\mathbf{\Gamma} \underset{\hookleftarrow}{\twoheadrightarrow} [0, 1]$
- Semicontinuous partial operations $-$ and $\sim$ on $\mathbf{\Gamma}$

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allow to define measures $X \to \mathbf{\Gamma}$

monotone functions $\mu\colon \mathrm{Clp}(X) \to \mathbf{\Gamma}$ s.t.

- $\mu(\emptyset) = 0^\circ$, $\mu(X) = 1^\circ$
- $\mu(A) \sim \mu(A \cap B) \leq \mu(A \cup B) - \mu(B)$
- $\mu(A) - \mu(A \cap b) \geq \mu(A \cup B) \sim \mu(B)$

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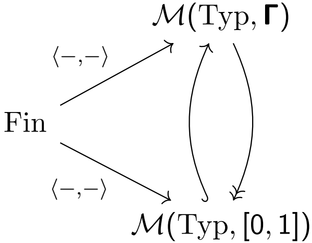
  allow to define measures $X \to \Gamma$
- $X \mapsto \mathcal{M}(X, \Gamma)$ endofunctor on Stone spaces
- We also have $\langle -, - \rangle : \mathrm{Fin} \to \mathcal{M}(\mathrm{Typ}, \Gamma)$ such that



Theorem [Gehrke, J., Reggio, 2019]**.**
If X is dual to B then $\mathcal{M}(X, \mathbf{\Gamma})$ is dual to $\mathbf{P}(B)$, the free Boolean algebra on the set of generators

$$\mathbb{P}_{\geq q}\,\varphi \qquad (\text{for } \varphi \in D, q \in [0,1] \cap \mathbb{Q})$$

and factored by the congruence $\models$ given below

Intuitively, $A \models \mathbb{P}_{\geq q}\,\varphi$ if
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(L1) $p \leq q$ implies $\mathbb{P}_{\geq q}\, \varphi \models \mathbb{P}_{\geq p}\, \varphi$

If the probability of $A \models \varphi(\overline{a})$ is $\geq q$ then it is also $\geq p$

(L2) $\varphi \leq \psi$ implies $\mathbb{P}_{\geq q}\, \varphi \models \mathbb{P}_{\geq q}\, \psi$

(L3) $\mathbb{P}_{\geq p}\, \mathbf{f} \models \mathbf{f}$ for $p > 0$, $\mathbf{t} \models \mathbb{P}_{\geq 0}\, \mathbf{f}$, and $\mathbf{t} \models \mathbb{P}_{\geq q}\, \mathbf{t}$

(L4) $0 \leq p + q - r \leq 1$ implies

$$\mathbb{P}_{\geq p+q-r}\, (\varphi \vee \psi) \wedge \mathbb{P}_{\geq r}\, (\varphi \wedge \psi) \models \mathbb{P}_{\geq p}\, \varphi \vee \mathbb{P}_{\geq q}\, \psi \quad \text{and}$$

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$$\mathbb{P}_{\geq p}\,\varphi \wedge \mathbb{P}_{\geq q}\,\psi \models \mathbb{P}_{\geq p+q-r}\,(\varphi \vee \psi) \vee \mathbb{P}_{\geq r}\,(\varphi \wedge \psi)$$

(Think of $\neg\mathbb{P}_{\geq p}\,\varphi$ as $\mathbb{P}_{<p}\,\varphi$)

Stone pairing logically

- Then $\mathcal{M}(\mathrm{Typ}, \mathbf{\Gamma})$ is the space of complete consistent theories for the logic of $\mathbf{P}(\mathrm{FO})$ (i.e. an extension of FO).

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$$\mathrm{Fin} \to \mathcal{M}(\mathrm{Typ}, \mathbf{\Gamma}), \quad A \mapsto \langle -, A \rangle$$

maps $A \in \mathrm{Fin}$ the theory containing

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- The space X in $\beta(\mathrm{Fin}) \twoheadrightarrow X \hookrightarrow \mathcal{M}(\mathrm{Typ}, \mathbf{\Gamma})$ is dual to $\mathbf{P}(\mathrm{FO})/_{\sim}$ where

$$\mathbb{P}_{\geq p}\, \varphi \sim \mathbb{P}_{\geq q}\, \psi \quad \text{iff} \quad \forall A \in \mathrm{Fin}\ \langle \varphi, A \rangle \geq p \leftrightarrow \langle \psi, A \rangle \geq q$$

Conclusion

Topological or duality theoretical techniques elsewhere:

- Duality-theoretical story in database theory (schema mappings)?
- Can duality theory say something interesting about $\mathbb{P}_k, \mathbb{E}_k, \mathbb{M}_k$?
- Topological approach to 0–1 laws?
- Logical approach to probabilistic powerdomains? Or replace [0,1] by $\mathbf{\Gamma}$ as the valuation space?
- Is the slogan "quantifiers $\longleftrightarrow$ measures" justified? More examples? Counterexamples?

Thank you!

(see arXiv:1907.04036 for details about probabilistic quantifiers)