Quantifiers and Measures, Part I

Luca Reggio

University of Oxford

Resources and Co-Resources workshop
March 26, 2020

- Probl: understand quantification as a semantic construction
- Suggest a connection between quantification and a space-of-measures construction
  - in logic on words (Gehrke, Petrişan, R. – ICALP16, LICS17)
  - in FMT (Gehrke, Jakl, R. – FOSSACS20)
- In this talk: the broader context in which these results can be understood

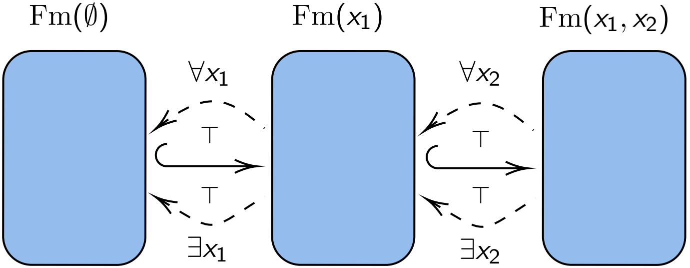
Algebras from logic

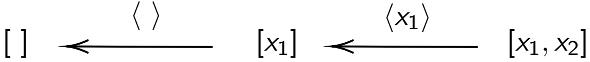
- Boole and Pierce (propositional setting) and later — at a more foundational level — Tarski, Lindenbaum and other algebraic logicians
- the idea is to obtain an algebra from a logic $\mathcal{L}$, called the Lindenbaum-Tarski algebra of $\mathcal{L}$, by quotienting the set of all formulas for $\mathcal{L}$ by logical equivalence.

CPL	$\leftrightarrow$	Boolean algebras
'positive' CPL	$\leftrightarrow$	distributive lattices
modal logics	$\leftrightarrow$	(varieties of) modal algebras

For logics with quantifiers, more ‘structure’ is needed:

- polyadic or cylindric algebras (Halmos, Henkin, Monk, Tarski)
  - quantifiers are added as an extra piece of structure
- Lawvere’s idea of quantifiers as adjoints to substitutions (presence of quantifiers as a property of certain morphisms) leading to hyperdoctrines and categorical logic





In general, a morphism $[x_1, \ldots, x_k] \to [x_1, \ldots, x_n]$ between contexts is a substitution $\langle t_1(x_1, \ldots, x_k), \ldots, t_n(x_1, \ldots, x_k)\rangle$.

Topological methods for logic

Developing and applying topological methods in logic is one of the main focuses of duality theory, which allows for a rigorous study of the connection between syntax and semantics.

- Stone (1936-38): dualities for Boolean algebras and distributive lattices
- Jónsson-Tarski (1951): duality for extended operations (canonical extensions)
- Goldblatt (1989): general topological treatment of duality for extended operations

For (classical) FO logic, dual spaces are 'easy': they are the spaces of models/types. Given a set of variables $V = \{x_1, x_2, \dots\}$ and a theory T in a signature σ, consider the sets

$$\mathrm{Mod}_\omega = \{(A, \alpha\colon V \to A) \mid A \text{ is a } \sigma\text{-structure and } A \models T\},$$
$$\mathrm{FO} = \{\text{first-order formulas in the signature } \sigma \text{ over the variables } V\}.$$

The satisfaction relation $\models \subseteq \mathrm{Mod}_\omega \times \mathrm{FO}$ induces the equivalence relations of elementary equivalence and logical equivalence on these sets, respectively:

$$(A, \alpha) \equiv (A', \alpha') \quad \text{iff} \quad \forall \varphi \in \mathrm{FO} \;\; A, \alpha \models \varphi \iff A', \alpha' \models \varphi,$$

$$\varphi \approx \psi \quad \text{iff} \quad \forall (A, \alpha) \in \mathrm{Mod}_\omega \;\; A, \alpha \models \varphi \iff A, \alpha \models \psi.$$

The quotient

$$\mathrm{FO}(T) = \mathrm{FO}/{\approx}$$

carries a natural Boolean algebra structure and is known as the Lindenbaum-Tarski algebra of T. On the other hand,

$$\mathrm{Typ}(T) = \mathrm{Mod}_\omega/{\equiv}$$

is naturally equipped with a topology, generated by the sets

$$[\![\varphi]\!] = \{[(A, \alpha)] \mid A, \alpha \models \varphi\}$$

for $\varphi \in \mathrm{FO}$, and is known as the space of types of T.
Gödel's completeness theorem may now be stated as the fact that

$\mathrm{Typ}(T)$ is the Stone dual space of $\mathrm{FO}(T)$.

- For other logics the dual spaces can be somewhat exotic, e.g. for the sentences of Büchi's logic on words over a given finite alphabet A, it is the free profinite monoid over A.
- In Domain Theory in Logical Form (Abramsky, 1991), the dual spaces are bifinite domains. The algebras providing the logic are a certain class of 'bifinite' distributive lattices.
- There are topological methods in logic which do not a priori originate from duality:
  - profinite methods in logic on words [GGP08],[GPR17]
  - structural limits in FMT [GJR20]
  - limit objects in database theory (e.g. Kolaitis' schema mappings) ???

Inductive vs co-inductive approaches

In the setting of categorical logic and hyperdoctrines, one builds the Lindenbaum-Tarski algebra starting from the sentences $\rightsquigarrow$ impredicative (/co-inductive?) approach.

We want to see quantification as a construction, and identify the corresponding dual effect of applying a layer of quantifiers. This problem has been (partially) addressed in several frameworks:

- $\exists/\Diamond \leftrightarrow$ Vietoris hyperspace (Johnstone '82, Abramsky '88,...)
- semiring/probability quantifiers $\leftrightarrow$ spaces of finitely additive measures (Gehrke, Jakl, Petrișan, R.)

We recall the link between $\exists$ and Vietoris for arbitrary structures.

Fix a theory T and let $\mathrm{FO}_n(T)$ be the algebra of (equivalence classes of) formulas with free variables among $x_1, \ldots, x_n$. If

$$\mathrm{Mod}_n = \{[(A, \alpha\colon \{x_1, \ldots, x_n\} \to A)] \mid A \text{ is a } \sigma\text{-structure and } A \models T\},$$

we obtain an embedding

$$\mathrm{FO}_n(T) \xhookrightarrow{[\![\ ]\!]_n} \mathcal{P}(\mathrm{Mod}_n), \ [\varphi] \mapsto [\![\varphi]\!]_n = \{[(A, \alpha)] \in \mathrm{Mod}_n \mid A, \alpha \models \varphi\}.$$

Now, consider the projection map

$$\pi\colon \mathrm{Mod}_n \twoheadrightarrow \mathrm{Mod}_{n-1}$$

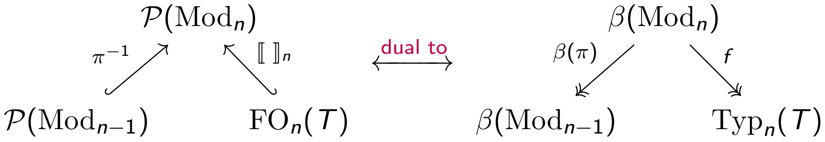
which forgets the last coordinate, and observe that

$$\pi([\![\varphi]\!]_n) = [\![\exists x_n.\varphi]\!]_{n-1}.$$

We define $B_{\exists x_n}$ as the Boolean subalgebra of $\mathcal{P}(\mathrm{Mod}_{n-1})$ generated by the set

$$\{\pi(\llbracket\varphi\rrbracket_n) \mid \varphi \in \mathrm{FO}_n(T)\} = \{\llbracket\exists x_n.\varphi\rrbracket_{n-1} \mid \varphi \in \mathrm{FO}_n(T)\}.$$

$B_{\exists x_n}$ is the Boolean algebra obtained by adding $\exists x_n$ to $\mathrm{FO}_n(T)$.



We get a relation on the space side:

$$R_f \colon \beta(\mathrm{Mod}_{n-1}) \nrightarrow \mathrm{Typ}_n(T), \quad x R_f y \iff y \in f(\beta(\pi)^{-1}(x)).$$

($f(\beta(\pi)^{-1}(x))$ is a closed set!)

We can regard $R_f \colon \beta(\mathrm{Mod}_{n-1}) \nrightarrow \mathrm{Typ}_n(T)$ as a function

$$R_f \colon \beta(\mathrm{Mod}_{n-1}) \to \mathcal{V}(\mathrm{Typ}_n(T))$$

where $\mathcal{V}(X)$ is the Vietoris hyperspace of the (Boolean) space X.

The elements of $\mathcal{V}(X)$ are the closed subsets of X, and the topology is generated by the sets

$$\Diamond U = \{C \in \mathcal{V}(X) \mid C \cap U \neq \emptyset\} \text{ and } (\Diamond U)^c$$

for every clopen subset U of X.

Proposition ($\exists$ vs Vietoris)

The map $R_f \colon \beta(\mathrm{Mod}_{n-1}) \to \mathcal{V}(\mathrm{Typ}_n(T))$ is continuous and its image is the dual Stone space of the Boolean algebra $B_{\exists x_n}$.

The completeness issue: how to characterise the continuous maps

$$R_f \colon \beta(\mathrm{Mod}_{n-1}) \to \mathcal{V}(\mathrm{Typ}_n(T))$$

which arise in this manner by 'quantification'?

In general, we do not know. In logic on words, thanks to the extra monoid structure (in the form of monoid actions) available, such continuous maps are characterised by a Reutenauer-type result as the length-preserving ones (Gehrke, Petrişan, R. – LICS17).

In the second part, Tomáš will show how $\mathcal{V}(X)$ can be seen as a space of two-valued measures on X and how this shift of perspective is useful when dealing with more general quantifiers.

Thank you for your attention!