# Concurrent Games over Relational Structures
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Propose integration of descriptive complexity with a general theory of games which supports resource.

**General reason:** to take advantage of a *resourceful* model based on concurrent games and strategies, developed and well tested in semantics; it supports the computational, logical, quantitative aspects, so resource as number of pebbles, degree of parallelism, probabilistic and quantum resource, ...

**Specific issues:** Oddities, limitations, in presenting strategies as coKleisli maps, homomorphisms $D(A) \to B$: bias to one-sided games ; composition of strategies = composition of coKleisli maps, is not obviously the usual composition of strategies! When is it so? Where do the comonads come from?

Thanks to: A. Ó Conghaile, S. Huriot-Tattegrain, Y. Montacute

# Games for interaction, via the composition of strategies

*In 2-party games read Player vs. Opponent as Process vs. Environment.*
*Follow the paradigm of Conway, Joyal to achieve compositionality.*

*Assume operations on (2-party) games:*

*Dual game* $G^\perp$ *- interchange the role of Player and Opponent;*
*Counter-strategy = strategy for Opponent = strategy for Player in dual game.*

*Parallel composition of games* $G \| H$.

*A strategy (for Player) from a game* $G$ *to a game* $H$ = *strategy in* $G^\perp \| H$.
*A strategy (for Player) from a game* $H$ *to a game* $K$ = *strategy in* $H^\perp \| K$.

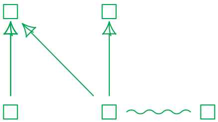
*Compose by letting them play against each other in the common game* $H$.

$\rightsquigarrow$ *a category with identity w.r.t. composition, the Copycat strategy in* $G^\perp \| G$,
*so from* $G$ *to* $G$ *...*

# Event structures - of the simplest kind

*An event structure comprises $(E, \leqslant, \#)$, consisting of a set of events $E$*
*- partially ordered by $\leqslant$, the causal dependency relation, and*
*- a binary irreflexive symmetric relation, the conflict relation,*
*which satisfy $\{e' \mid e' \leqslant e\}$ is finite and $e \# e' \leqslant e'' \implies e \# e''$.*

*Two events are concurrent when neither in conflict nor causally related.*



(drawn immediate conflict, and causal dependency)

*The configurations of an event structure $E$ consist of those subsets $x \subseteq E$ which are*
Consistent: $\forall e, e' \in x.\ \neg(e \# e')$ *and*
Down-closed: $\forall e, e'.\ e' \leqslant e \in x \implies e' \in x$.

# Event-structure game w.r.t. a signature

*A signature $(\Sigma, C, V)$ comprises $\Sigma$ a many-sorted relational signature including equality; a set $C$ event-name constants; a set $V = \{\alpha, \beta, \gamma, \cdots\}$ of variables.*

*A $(\Sigma, C, V)$-signature game comprises an event structure $(E, \leqslant, \#)$ – its moves are the events $E$, with*

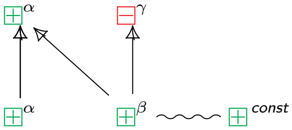
*a polarity function $\mathrm{pol} : E \to \{+, -\}$ s.t. no immediate conflict $\boxplus \sim\!\sim\!\sim \boxminus$*

*a variable/constant assignment $\mathrm{var} : E \to C \cup V$ s.t.*

$$e \text{ co } e' \Rightarrow \mathrm{var}(e) \neq \mathrm{var}(e')$$

*a winning condition $WC$, an assertion in the free logic over $(\Sigma, C, V)$.*

WC:
$\mathbb{E}(\gamma) \to \exists \beta.\ P(\alpha, \beta) \wedge Q(\beta)$
Existence predicate involves latest occurrence of variable in a configuration

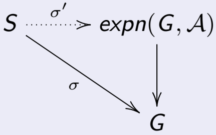



A good reference for free logic: Dana Scott, Identity and Existence. LNM 753, 1979

# Games over a structure

*A* *game over a structure* $(G, \mathcal{A})$ *is a* $(\Sigma, C, V)$*-game* $G$ *and* $\Sigma$*-structure* $\mathcal{A}$*.*
*It determines a (traditional) concurrent game* $expn(G, \mathcal{A})$ *in which each move with a variable* $\square^{\alpha}$ *is expanded to its instances* $\square^{a_1} \rightsquigarrow \square^{a_2} \rightsquigarrow \cdots$

*A* *strategy* $(\sigma, \rho)$ *in* $(G, \mathcal{A})$ *assigns values in* $\mathcal{A}$ *to Player moves of the game* $G$ *in answer to assignments of Opponent. Described as a map of event structures, it corresponds to a (traditional) concurrent strategy* $\sigma'$ *in* $expn(G, \mathcal{A})$*:*




*For a configuration* $x$ *of* $S$ *and a* $\Sigma$*-assertion* $\varphi$*,*
$x \models \varphi$ *will mean latest assignments to variables in* $x$ *make* $\varphi$ *true.*
*The strategy is* *winning* *means* $x \models WC$ *for all* $+$*-maximal configs* $x$ *of* $S$*.*

**Proposition.** The events $S$ of a strategy form a $\Sigma$-structure:

$$R_S(s_1, \cdots, s_n) \text{ iff } x \models R(\mathrm{var}(\sigma(s_1)), \cdots, \mathrm{var}(\sigma(s_n)))\,,$$

for some configuration $x$ of $S$ with $s_1, \cdots, s_n \in x$.

**Corollary.** $(G, \mathcal{A})$ determines a $\Sigma$-structure, on $V$-events $expn(G, \mathcal{A})_V$.
It extends to a comonad over $\Sigma$-structures.

Event strs. provide the interaction shapes with which to build comonads!

# Constructions on signature games

*Let $G$ be a $(\Sigma, C, V)$-game. Its dual $G^{\perp}$ is the $(\Sigma, C, V)$-game obtained by reversing polarities, i.e. the roles of Player and Opponent, with winning condition $\neg WC_G$.*

*Let $G$ be a $(\Sigma_G, C_G, V_G)$-game. Let $H$ be a $(\Sigma_H, C_H, V_H)$-game. Their parallel composition $G \| H$ is the $(\Sigma_G + \Sigma_H, C_G + C_H, V_G + V_H)$-game comprising the parallel juxtaposition of event structures with winning condition $WC_G \vee WC_H$.*

*Let $(G, \mathcal{A})$ to $(H, \mathcal{B})$ be games over structures. A winning strategy from $(G, \mathcal{A})$ to $(H, \mathcal{B})$ comprises a winning strategy in the game $(G^{\perp} \| H, \mathcal{A} + \mathcal{B})$ - its winning condition is $WC_G \rightarrow WC_H$.*

**Theorem.** Obtain a (bi)category of winning strategies between games over structures: winning strategies compose with the copycat strategy as identity.

Its maps are reductions: a winning strategy $\sigma$ from $(G, \mathcal{A})$ to $(H, \mathcal{B})$ reduces the problem of finding a winning strategy in $(H, \mathcal{B})$ to finding a winning strategy in $(G, \mathcal{A})$. A winning strategy in $(G, \mathcal{A})$ is a winning strategy from $(\varnothing, \varnothing)$ to $(G, \mathcal{A})$; its composition with $\sigma$ is a winning strategy in $(H, \mathcal{B})$.

# Spoiler-Duplicator games deconstructed

*A Spoiler-Duplicator game is specified by a deterministic concurrent strategy*

$$\begin{array}{c} D \\ \Big\downarrow{\scriptstyle \delta} \\ G^{\perp} \| G \end{array}$$

*which is an idempotent comonad $\delta$ in the bicategory of signature games.*
*Idea: $D$, itself a signature game, specifies the pattern of strategies from $(G, \mathcal{A})$ to $(G, \mathcal{B})$, whether they follow copycat, are all-in-one, ...*

*The* Spoiler-Duplicator category $\mathrm{SD}_{\delta}$ *has maps $(\sigma, \rho) : \mathcal{A} \rightarrow\!\!\!\!\!+\!\!> _{\delta} \mathcal{B}$ those deterministic strategies $(\sigma, \rho)$ from $(G, \mathcal{A})$ to $(G, \mathcal{B})$ which factor openly through $\delta$, i.e. so*

$$\begin{array}{ccc} S & \overset{open}{\dashrightarrow} & D \\ & {\scriptstyle \sigma}\searrow & \Big\downarrow{\scriptstyle \delta} \\ & & G^{\perp} \| G . \end{array}$$

# Characterising $\mathrm{SD}_\delta$ (for $\delta : D \to G^\perp \| G$)

*Assume $G$ has signature $(\Sigma, V, C)$. For $\Sigma$-structures $\mathcal{A}$ and $\mathcal{B}$, define the partial expansion $expn^-(D, \mathcal{A} + \mathcal{B})$ w.r.t. just Opponent moves. Define $D(\mathcal{A}, \mathcal{B})$ to be the set of its Player $V$-moves.*

Strategies $\mathcal{A} \twoheadrightarrow_\delta \mathcal{B}$ in $\mathrm{SD}_\delta$ correspond to functions

$$h : D(\mathcal{A}, \mathcal{B}) \to \mathcal{A} + \mathcal{B}$$

assigning elements of $\mathcal{A}$ and $\mathcal{B}$ to $V$-moves of Player. Composition à la GoI.

Assume $G$ is one-sided, *i.e.* all its $V$-moves are of Player. Then,

$$h : D(\mathcal{A}) \to \mathcal{B}\,.$$

It has a coextension $h^\dagger : D(\mathcal{A}) \to D(\mathcal{B})$ (relies on the idempotence of $\delta$ ).

Strategies $\mathcal{A} \twoheadrightarrow_\delta \mathcal{B}$ in $\mathrm{SD}_\delta$ correspond to $h : D(\mathcal{A}) \to \mathcal{B}$ which preserve winning conditions $W_G$ across $+$-maximal configurations of $D$; they compose via coextension.

# Strategies as coKleisli maps

*$D(\mathcal{A})$ inherits $\Sigma$-structure from $\mathcal{A}$ — via the counit of $\delta$ each Player $V$-move $e$ depends on an earlier corresponding assignment $\bar{e}$ of Opponent:*
*$R(e_1, \cdots, e_k)$ in $D(\mathcal{A})$ iff $x \models R(\bar{e}_1, \cdots, \bar{e}_k)$, some $+$-maxl config $x$ of $D(\mathcal{A})$.*
*Coextension preserves homomorphisms; $D(\_)$ a comonad on $\Sigma$-structures.*

*When $G$ is one-sided and $\delta$ is copycat, the comonad $D(\_)$ is isomorphic to that of $expn(G, \_)_V$ on earlier slide — cf. SmP 2021 talk.*

Often, depending on the winning conditions $W_G$, the coKleisli category of $D(\_)$ is isomorphic to $\mathrm{SD}_\delta$, for example in these cases:

for game $G$ and $\delta$ as copycat for pebbling comonads [Abramsky, Dawar, Wang]

for game $G$ and $\delta$ as copycat for simulation [Abramsky, Shah]

for game $G$ and $\delta$ enforcing delay for all-in-one game for trace inclusion

for game $G$ and $\delta$ enforcing delay for all-in-one game of the pebble-relation comonad [Montacute, Shah]

# Examples: the $k$-pebble game and simulation game

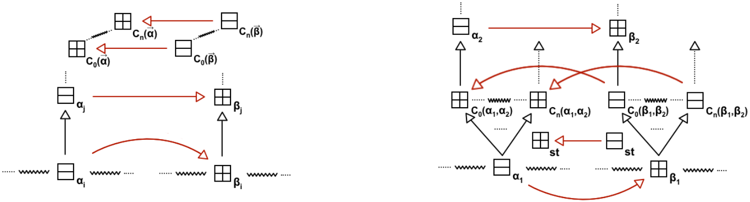



Figure: the $k$-pebble game (left) and the simulation game (right).

The $k$-pebble game $\delta_0 : ⅅ_{G_0} \to G_0^{\perp} \| G_0$ with

$$W_{G_0} \equiv \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\vec{\beta})) \to R_i(\vec{\beta}) \, .$$

The simulation game $\delta_1 : ⅅ_{G_1} \to G_1^{\perp} \| G_1$ with

$$W_{G_1} \equiv \mathbb{E}(st) \to Start(\beta_1) \ \wedge$$
$$\bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\beta_1, \beta_2)) \to R_i(\beta_1, \beta_2) \ \wedge \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\beta_2, \beta_1)) \to R_i(\beta_2, \beta_1) \, .$$

# Example: the trace-inclusion game

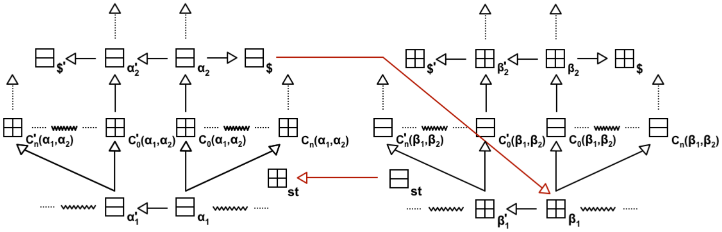
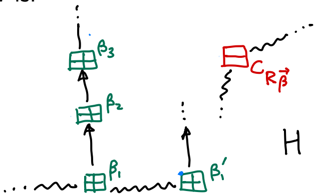



Figure: The trace-inclusion game

The trace-inclusion game $\delta_2 : D \to G_2^{\perp} \| G_2$ with

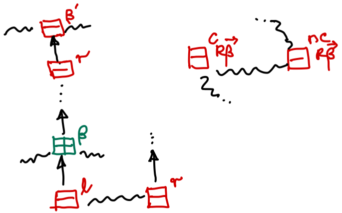
$$
\begin{aligned}
W_{G_2} \equiv W_{G_1} \;\wedge & \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i'(\beta_1, \beta_2)) \to R_i'(\beta_1', \beta_2') \\
\wedge & \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i'(\beta_2, \beta_1)) \to R_i'(\beta_2', \beta_1') \\
\wedge & \; (\mathbb{E}(\beta_1') \to \beta_1 \coloneq\!\!< \beta_1') \wedge (\mathbb{E}(\beta_2') \to \beta_2 \coloneq\!\!< \beta_2') \wedge \mathbb{E}(\$) \wedge \mathbb{E}(\$').
\end{aligned}
$$

Example, **Homomorphism game** $\mathrm{SD}_{\propto_H}$ where $H$ is:




with winning condition $W_H \equiv \bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \to R(\vec{\beta})$ where $\vec{\beta}$ is a tuple of variables.

Example, **Ehrenfeucht-Fraïssé games** $\mathrm{SD}_{\propto_{EF}}$ where $EF$ is:



with winning condition

$W_{EF} \equiv (\bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \to R(\vec{\beta})) \wedge (\bigwedge_{R\vec{\beta}} \mathbb{E}(nc_{R\vec{\beta}}) \to \neg R(\vec{\beta}))$.

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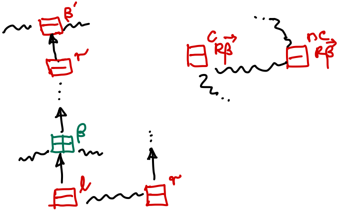

with winning condition
$W_{EF} \equiv (\bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \rightarrow R(\vec{\beta})) \wedge (\bigwedge_{R\vec{\beta}} \mathbb{E}(nc_{R\vec{\beta}}) \rightarrow \neg R(\vec{\beta}))$.